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MATHEMATICAL FOUNDATIONS OF LIMIT CRITERION FOR ANISOTROPIC MATERIALS

MATEMATYCZNE PODSTAWY WARUNKU GRANICZNEGO DLA MATERIAŁÓW ANIZOTROPOWYCH

In the paper a new proposition of limit state criteria for anisotropic solids exhibiting different strengths at tension and compression is presented. The proposition is based on the concept of energetically orthogonal decompositions of stress state introduced by Rychlewski. The concept of stress state dependent parameters describing the influence of certain stress modes on the total measure of material effort was firstly presented by Burzyński. The both concepts are reviewed in the paper. General formulation of a new limit criterion as well as its specification for certain elastic symmetries is given. It is compared with some of the other known limit criteria for anisotropic solids. General methodology of acquiring necessary data for the criterion specification is presented. The ideas of energetic and limit state orthogonality are discussed – their application in representation of the quadratic forms of energy and limit state criterion as a sum of square terms is shown.

Keywords: anisotropy, strength hypothesis, material e?ort, yield condition, plastic potential, strength differential effect, elasticity, plasticity

Artykuł przedstawia nową propozycję kryterium stanu granicznego dla anizotropowych ciał stałych wykazujących różną wytrzymałość przy rozciąganiu i ściskaniu. Propozycja ta opiera się na koncepcji energetycznie ortogonalnych rozkładów stanu naprężenia zaproponowanej przez Rychlewskiego. Pomysł wykorzystania zależnych od stanu naprężenia parametrów określających wpływ konkretnych form naprężenia na całkowitą miarę wytężenia materiału po raz pierwszy został przedstawiony przez Burzyńskiego. W pracy omówione są obydwie koncepcje. Przedstawione jest ogólne sformułowanie nowego warunku granicznego, jak również jego specyfikacja dla wybranych symetrii sprężystych. Kryterium to zostało porównane z niektórymi innymi spośród znanych kryteriów granicznych dla ciał anizotropowych. Przedstawiono ogólną metodologię pozyskiwania danych wymaganych do specyfikacji kryterium.

1. Introduction

1.1. Motivation

Rapid development of materials science is related with increasing demand for adequate analysis of deformation processes in newly elaborated materials. Also the simulations of thermomechanical processes that appear in many recent industrial and laboratory applications require accounting for some unconventional mechanical properties of investigated solids. Among the uncommon properties of materials, which cannot be neglected any more, one should distinguish especially anisotropy of mechanical properties, tailored often on demand by controlling the structure on the microscopic or nano-metric level, and asymmetry of elastic range (cf. e.g. the discussion of the limit states in nano-metals by Frąś et al. [1]). The latter property can be observed as the difference of tensile and compression strength – the so called strength differential effect (SDE), pressure sensitivity of yield and the dependency on the third invariant of stress deviator described by the so

called Lode angle on the octahedral plane. It is the goal of modern mechanics of solids to provide such a mathematical description, which enables to account for these features of modern materials. The aim of the paper is to introduce a new mathematical formulation of yield criteria for anisotropic solids revealing the asymmetry of elastic range. The novelty of the proposed approach is based on the hypothesis that certain parts of elastic energy density can be applied to formulate the measure of material effort. The notion of material effort, known since long in German literature on mechanics of solids as *Anstrengung*, in Russian напрыженность (*napryazhen-nost'*) and in Polish as *wytężenie*, used rather intuitively, can be defined more precisely as the state of material point of the loaded body, determined by internal forces (stress) and strain, which is related with the change of the strength of chemical bonds, with respect to the natural state, in the representative volume element of the condensed matter under investigation. This definition is in accord with the earlier studies about the relation of microscopic observations and modelling of deformation and fracture of solids by Pęcherski [2]. The interdisci-

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plinary approach of connecting the nature of chemical bonds with the strength of materials was presented by Gilman [3]. The applications of Molecular Dynamics for the analysis of the strength of chemical bonds determining the strength of metal-metal oxide interfaces were studied in the papers of Nalepka and Pęcherski [4], [5] as well as in Nalepka [6], [7]. A measure of material effort is required to assess the distance of the considered state of stress from the postulated surface of limit states in the six-dimensional space of stress. The conclusion of the aforementioned studies is that the notion of energy as a multilevel scalar quantity can be applied as the universal and versatile measure of material effort. On the atomic level certain part of energy is related with the change of the strength of chemical bonds. Similarly, on the macroscopic level some precisely defined parts (contributions) of the density of elastic energy accumulated in the strained body contribute to the measure of material effort. The symmetry of the elastic and strength properties control the particular partition of the total elastic energy density. Historically, the first proposition of accounting for the density of elastic energy as a measure of material effort in isotropic solids belongs to James Clerk Maxwell, who expressed this idea in the private letter of the 18th December 1856 to William Thomson (known later as Lord Kelvin). The letter has been published with other works of Maxwell in 1936 [8]. Independently, Beltrami [9] considered the density of total elastic energy as a measure of material effort, what has not found conformation in experimental investigations. More successful approach was presented by Huber in 1904[10] who assumed on the basis of certain physical reasoning of molecular interactions that the density of elastic energy of distortion is an appropriate measure of material effort for isotropic solids. The earlier proposition of Maxwell supports this idea. The mentioned partition of energy density for anisotropic solids was first introduced by William Thomson [11] in 1856yet it has not become a subject of broader interest until Jan Rychlewski undertaken systematic study of this problem [12], [13]. The all discussed above cases of the energy-based hypothesis of material effort are related with the fundamental assumption about the symmetry of elastic range. In terms of the measured elastic limits (yield strengths) it means that their values in tension and compression are the same. If the distinct difference of these values is observed, the asymmetry of elastic range appears and the formulation of energy-based hypothesis of material effort for anisotropic solids remains an open and unsolved problem. The complete and mathematically elegant solution for the case of isotropic body was provided by Burzyński [14].

1.2. Brief review on the limit criteria for anisotropic solids

In 1928 Richard von Mises in his pivotal paper [15] has presented at least three innovative general ideas related with limit criteria for anisotropic solids. First, the attention was focused on the concept of a stress state dependent quadratic function satisfying the postulates of symmetry preservation and pressure insensitivity – it was the extension of the limit state criterion for isotropic solids presented in 1913 [16], anticipated by Huber as early as in 1904 [10]. Von Mises' idea was the basis for series of fur-

ther propositions dealing with different elastic symmetries or involving the influence of hydrostatic stress. The best known are the limit criteria by Hill [17] and Hoffman [18] as well as by Tsai-Wu [19], Caddell-Raghava-Atkins [20] and Deshpande-Fleck-Ashby [21]. The idea of quadratic functions was soon modified so that the exponent of the terms of the considered function was fixed at different value (eg. Liu-Huang-Stout [22]) or it became another parameter of the criterion (eg. Hill [23] and Hosford [24]). Basing on those concepts a large number of yield criteria for plane stress state were formulated – for precise summary and analysis see e.g. Banabic [25]. Most of the discussed limit criteria cannot describe any symmetry lower than orthotropy, some cannot account for the strength differential effect and sometimes the obtained yield surface simply does not correspond well with the experimental results. The ways of dealing with this problem were also of purely technical nature – adding further linear or quadratic terms involving additional constant parameters [18] [19] [20] [21], changing the value of exponent [23], [24] etc. In the authors' opinion this is not a proper way of solving the problem – the results are complex and unclear in their mathematical form, which is even more difficult for both mathematical analysis and physical interpretation.

The second idea presented in the paper of Mises [15] was – as it seems – the first attempt of generalization of the Coulomb-Tresca-Guest limit shear stress condition. Another such proposition was suggested by Hu [26]. The third idea of the paper of Mises [15]is based on very general (yet never developed) proposition of a plasticity condition. Mises suggested that the most general yield condition should be a function which arguments are certain stress state invariants with respect to the addition of the hydrostatic stress state and the geometrical transformations belonging to the group of symmetry of the considered material. He has also proposed a "complete set" of such invariants for cubic and hexagonal symmetry. Valuable proposition of a yield criterion for anisotropic solids was also made by Karafillis and Boyce [27]. The authors where considering "isotropic plasticity equivalent" (IPE) yield surface which was obtained with use of a properly chosen (with respect to the symmetries of the elasticity tensors) linear transformation of the stress space. Quite recently an interesting proposition of Schreyer and Zuowas presented [28],in which the spectral decomposition of the elasticity tensors was used for definition of a set of the yield criteria and a set of uncoupled equations for the evolution parameters respective for all eigenstates of the elasticity tensors. Also recently the geometrical foundations of plasticity yield criteria were studied with use of differential geometry and group theory concepts [29]. Yet still certain required features such as accounting for the SD effect, simplicity in application or clear physical interpretation are missing in those propositions.

The last group of the limit state criteria is the one which considers as a measure of material effort certain parts of stored elastic energy density. An interesting coincidence is the fact that the first energy-based hypothesis of material effort for anisotropic solids was stated by Burzyński [14] in the same year when Mises published his paper [15]. Burzyński considered a special class of materials for which the decomposition of the elastic energy into its volumetric and distortional part is possible – the combination of those energies, influ-

enced additionally by the hydrostatic stress was considered as a measure of material effort. Yet Burzyński has not provided any workable criterion for anisotropic solids. He focused rather on certain modification of the criterion for the isotropic case accounting for the effect of initial anisotropy [30]. In 1956 Olszak and Urbanowski [31] inspired by earlier works of Goldenblat [32] [33] introduced a limit condition for anisotropic solids involving a "pseudoenergy" of distortion and volume change, however yet these notions still had no clear physical meaning. Then the concept – already with use of the spectral decomposition of elasticity tensors introduced by Rychlewski [12] – was further developed by Olszak and Ostrowska-Maciejewska in [34]. Yet the most general, and simultaneously physically strict and mathematically elegant was the proposition of Rychlewski [13], who also gave clear energetic interpretation of the generalized quadratic limit condition for anisotropic solids by Mises [15]. The detail discussion of the energy-based condition for cubic and transverse symmetries was given in [35].

As it was stated before, one has to notice that large number of all of the limit criteria mentioned above has certain drawbacks. Many of them were developed under the assumption that the hydrostatic stress does not influence neither the elasticity limit nor the plastic flow. Facing the experimental data gathered for decades this assumption cannot always hold true.A definite influence of pressure on the yield stress was proved by experiments (Spitzig, Richmond and Sober [36] [37], Wilson [38] and others). The relation between pressure sensitivity and the strength differential effect was also pointed out [39]. Another issue which seems to be almost totally omitted in the investigations of the discussed problem is that only single proposition (e.g. Cazacu and Barlat [40]) of the yield criteria for anisotropic solids account for the influence of the third invariant of the stress tensor deviator and thus – in particular – they are capable of describing the materials exhibiting the Lode angle dependence. Of course both third invariant of the stress tensor deviator and the Lode angle are invariants with respect to all possible rotations soit would be inconsistent to consider them as a variable influencing an anisotropic function directly without any restrictions being a consequence of the material's symmetry. Anyway, it is rather obvious if such a specific phenomenon as the Lode angle dependency can be observed in case of isotropic solids, similar, or even more sophisticated effects should occur in case of anisotropy. All those facts are the main reasons and a direct motivation for the authors to formulate a new proposition of a limit state criterion for anisotropic solids that would satisfy all of the above mentioned requirements.

1.3. The content of the paper

The second section of the paper is devoted to the derivation of the new proposition of an energy-based limit state criterion for anisotropic solids. Fundamental theorems on the spectral decomposition of a linear operator and on simultaneous representation of two quadratic forms in their canonical forms are given, as well as application and its consequences in certain problems of mathematical theory of plasticity are discussed. Previously introduced concept of energetic orthogonality [13] as well as the new concept of the limit state orthogonality are presented. General idea of elastic energy decompositions of Rychlewski [13], and the idea of stress state dependent influence functions by Burzyński [14] are also described. The third section of the paper contains the statement of the authors' new proposition – general formulation, basic assumptions on the introduced quantities and discussion on the application of the presented yield criterion in the description of plastic deformation. In the fourth section exemplary specifications of the limit criteria for certain chosen elastic symmetries are given. The methodology of determination of the form of the unknown functions is also proposed. The paper closes with a brief summary.

2. Derivation

Consider linear constitutive relations between stress and strain (generalized Hooke's law):

$$
\begin{cases}
\boldsymbol{\sigma} = \mathbf{S} \cdot \boldsymbol{\varepsilon} \\
\boldsymbol{\varepsilon} = \mathbf{C} \cdot \boldsymbol{\sigma} \\
\mathbf{C} \circ \mathbf{S} = \mathbf{S} \circ \mathbf{C} = \mathbf{I}^S
\end{cases}
\Rightarrow
\begin{cases}
\sigma_{ij} = S_{ijkl}\varepsilon_{kl} \\
\varepsilon_{ij} = C_{ijkl}\sigma_{kl} \\
C_{ijkl}S_{klmn} = S_{ijkl}C_{klmn} = \\
\quad = \frac{1}{2}\left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right)
\end{cases},
$$
(1)

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor, $\mathbf{S}$ and $\mathbf{C}$ are symmetric, positive definite fourth rank elasticity tensors (stiffness and compliance tensor respectively) and $\mathbf{I}^S$ is the identity operator in the six-dimensional linear space of symmetric second order tensors $\mathcal{T}^2_{sym}$. Elasticity tensors satisfy the following internal symmetries:

$$
\begin{aligned}
S_{ijkl} &= S_{jikl} = S_{ijlk} = S_{klij}, \\
C_{ijkl} &= C_{jikl} = C_{ijlk} = C_{klij}.
\end{aligned}
$$
(2)

If the stress state space is considered as a dimensionless one (e.g. relative stresses referred to a fixed value of stress) then the generalized Hooke'slaw (1) can be considered as a linear map of $\mathcal{T}^2_{sym}$ onto itself – an automorphism in $\mathcal{T}^2_{sym}$.

2.1. Spectral decomposition of elasticity tensors

For any tensor of an even order $\mathbf{T} \in \mathcal{T}^{2p}$ its eigenvalue and eigentensor problem can be considered.Tensor $\mathbf{T}$ is then a linear operator mapping the linear space $\mathcal{T}^p$ into itself. The clue concept of the currently presented proposition is the spectral decomposition of the elasticity tensors. As the elasticity tensors are the linear operators realizing the linear automorphic map of the generalized Hooke's law in $\mathcal{T}^2_{sym}$ (the dimensionless stress and strain space), their eigenproblem can be stated, namely the problem of finding such symmetric second order tensor ω for which

$$
\mathbf{S} \cdot \omega = \lambda\omega \Rightarrow \left(\mathbf{S} - \lambda\mathbf{I}^S\right) \cdot \omega = \mathbf{0}.
$$
(3)

We call the scalar λ (the eigenvalue od $\mathbf{S}$) the Kelvin modulus [12] and the corresponding tensor ω is termed a proper state of $\mathbf{S}$. It is worth noting that the eigenproblem of the elasticity tensors is related to the problem of finding such strain state of fixed norm for which the stored elastic energy density reaches its local extremum

$$
\Phi = \frac{1}{2}\boldsymbol{\varepsilon} \cdot \boldsymbol{\sigma} = \frac{1}{2}\boldsymbol{\varepsilon} \cdot (\mathbf{S} \cdot \boldsymbol{\varepsilon}) \to \min/\max \ \Lambda |\boldsymbol{\varepsilon}| = \text{const.}
$$
(4)

Since the elasticity tensors are symmetric, all Kelvin moduli are real. Positive definiteness of $\mathbf{S}$ and $\mathbf{C}$ is equivalent to the positiveness of all Kelvin moduli. Since the compliance tensor is the inverse of the stiffness tensor, it can be shown that

$$\boldsymbol{\omega} = \mathbf{I}^S \cdot \boldsymbol{\omega} = (\mathbf{C} \circ \mathbf{S}) \cdot \boldsymbol{\omega} = \mathbf{C} \cdot (\mathbf{S} \cdot \boldsymbol{\omega}) = \mathbf{C} \cdot (\lambda \boldsymbol{\omega}) \quad (5)$$

what yields

$$\mathbf{C} \cdot \boldsymbol{\omega} = \frac{1}{\lambda} \boldsymbol{\omega}. \quad (6)$$

It is now visible that any eigenstate of $\mathbf{S}$ is also an eigenstate of $\mathbf{C}$ and the corresponding eigenvalue of $\mathbf{C}$ is the inversion of the pertinent Kelvin modulus. Rychlewski has introduced and proved the following theorem [12], which he called *the main structural formula of linear theory of elasticity*

Theorem

For any elastic body given by its stiffness tensor $\mathbf{S}$ there exists exactly one decomposition of the linear space of symmetric second order tensors $\mathcal{J}_{sym}^2$ into direct sum of mutually orthogonal subspaces $\mathcal{P}_\alpha$ (eigensubspaces of the elasticity tensors)

$$\begin{aligned} \mathcal{J}_{sym}^2 &= \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \dots \oplus \mathcal{P}_p, \quad \rho \leqslant 6 \\ \mathcal{P}_\alpha &\perp \mathcal{P}_\beta \text{ for } \alpha \neq \beta \quad \alpha, \beta = 1, 2, ..., \rho \end{aligned} \quad (7)$$

such that for any stress or strain state

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 + \dots + \boldsymbol{\sigma}_\rho, \quad \boldsymbol{\sigma}_\alpha \in \mathcal{P}_\alpha \\ \boldsymbol{\varepsilon} &= \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2 + \dots + \boldsymbol{\varepsilon}_\rho, \quad \boldsymbol{\varepsilon}_\alpha \in \mathcal{P}_\alpha \end{aligned} \quad (8)$$

and there exists exactly one set of pairwise unequal constants (Kelvin moduli – eigenvalues of the stiffness tensor)

$$\lambda_1, \lambda_2, \dots, \lambda_\rho, \quad \lambda_\alpha \neq \lambda_\beta \text{ for } \alpha \neq \beta$$

such that

$$\begin{aligned} \mathbf{S} &= \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 + \dots + \lambda_\rho \mathbf{P}_\rho \\ \mathbf{C} &= \frac{1}{\lambda_1} \mathbf{P}_1 + \frac{1}{\lambda_2} \mathbf{P}_2 + \dots + \frac{1}{\lambda_\rho} \mathbf{P}_\rho \end{aligned} \quad (9)$$

where linear operators $\mathbf{P}_\alpha$ are the orthogonal projectors on the eigensubspace $\mathcal{P}_\alpha$ corresponding with the λ_α eigenvalue:

$$\mathbf{P}_\alpha \cdot \boldsymbol{\sigma} = \begin{cases} \boldsymbol{\sigma} \Leftrightarrow \boldsymbol{\sigma} \in \mathcal{P}_\alpha \\ 0 \Leftrightarrow \boldsymbol{\sigma} \notin \mathcal{P}_\alpha \end{cases}. \quad (10)$$

The orthogonal projector $\mathbf{P}_\alpha$ are the identity operators in the corresponding eigensubspace $\mathcal{P}_\alpha$. Since the decomposition of $\mathcal{J}_{sym}^2$ into eigensubspaces of the elasticity tensors is complete, so

$$\mathbf{P}_1 + \mathbf{P}_2 + \dots \mathbf{P}_\rho = \mathbf{I}^S. \quad (11)$$

As the theorem states this decomposition is unique. However, any orthogonal projector $\mathbf{P}_\alpha$ itself can be expressed as a sum of dyads of the normalized eigenstates ω_α^i $(i = 1, 2, \dots, \dim P_\alpha)$, $\omega_\alpha^i \cdot \omega_\alpha^j = \delta_{ij}$ respective for λ_α forming an orthonormal basis in $\mathcal{P}_\alpha$

$$\mathbf{P}_\alpha = \sum_{i=1}^{\dim \mathcal{P}_\alpha} \left(\omega_\alpha^i \otimes \omega_\alpha^i \right), \quad \text{(no summation over } \alpha). \quad (12)$$

It should be reminded that for real eigenvalues their algebraic and geometric multiplicity are equal – the multiplicity of an eigenvalue as a root of the characteristic polynomial is equal the dimension of the corresponding eigensubspace.

In general the spectral decomposition of the elasticity tensors can be written in a following form

$$\begin{aligned} \mathbf{S} &= \lambda_I \left(\omega_I \otimes \omega_I \right) + \lambda_{II} \left(\omega_{II} \otimes \omega_{II} \right) + \dots + \lambda_{VI} \left(\omega_{IV} \otimes \omega_{IV} \right) \\ \mathbf{C} &= \frac{1}{\lambda_I} \left(\omega_I \otimes \omega_I \right) + \frac{1}{\lambda_{II}} \left(\omega_{II} \otimes \omega_{II} \right) + \dots + \frac{1}{\lambda_{VI}} \left(\omega_{VI} \otimes \omega_{VI} \right) \end{aligned}, \quad (13)$$

which is unique for different Kelvin moduli. If some of the Kelvin moduli are multiple, some of $\lambda_K (K = I, II, ..., VI)$ are equal and some of ω_K can be chosen in an infinite number of ways. Please note that such decomposition is not unique. Any stress and strain state can be decomposed in any orthonormal basis in $\mathcal{J}_{sym}^2$ formed by eigenstates ω_K

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{\sigma}_I + \boldsymbol{\sigma}_{II} + \dots + \boldsymbol{\sigma}_{VI}, & \boldsymbol{\sigma}_K &= (\boldsymbol{\sigma} \cdot \omega_K) \omega_K \\ & & &\text{(no summation over } K) \\ \boldsymbol{\varepsilon} &= \boldsymbol{\varepsilon}_I + \boldsymbol{\varepsilon}_{II} + \dots + \boldsymbol{\varepsilon}_{VI}, & \boldsymbol{\varepsilon}_K &= (\boldsymbol{\varepsilon} \cdot \omega_K) \omega_K \end{aligned} \quad (14)$$

Similar considerations were presented later independently by Mehrabadi and Cowin [41] and Sutcliffe [42].

2.1.1. Decoupled stress-strain constitutive relations

The consequences of decomposition of the stress/strain space into the eigensubspaces of the elasticity tensors are of great meaning. If the stress and strain tensors are decomposed in the basis of the eigenstates of the elasticity tensors, then the constitutive relations (1) becomes a set of uncoupled equations – the corresponding proper stress and strain states are strictly proportional [12], [43]

$$\boldsymbol{\sigma}_K = \lambda_K \boldsymbol{\varepsilon}_K \Leftrightarrow \boldsymbol{\varepsilon}_K = \frac{1}{\lambda_K} \boldsymbol{\sigma}_K \quad K = I, \dots, VI \\ \text{(no summation over } K). \quad (15)$$

It should be mentioned that historically the first use of uncoupled constitutive relations was introduced by Thomson [11] in the expression of the elastic energy density as a sum of squares of strains:

$$\Phi = \lambda_I \varepsilon_I^2 + \lambda_{II} \varepsilon_{II}^2 + \lambda_{III} \varepsilon_{III}^2 + \lambda_{IV} \varepsilon_{IV}^2 + \lambda_V \varepsilon_V^2 + \lambda_{VI} \varepsilon_{VI}^2 \quad (16)$$

2.1.2. Decoupled stress-strain constitutive relations

Since the decomposition (7) of $\mathcal{J}_{sym}^2$ into eigensubspaces of the elasticity tensors is orthogonal it can be easily shown that the work performed by a proper stress state be longing to a certain eigensubspace on the proper strain state belonging to different eigensubspace is zero

$$\alpha \neq \beta \Rightarrow \mathcal{L}\left(\boldsymbol{\sigma}_\alpha, \boldsymbol{\varepsilon}_\beta \right) = \frac{1}{2} \boldsymbol{\sigma}_\alpha \cdot \boldsymbol{\varepsilon}_\beta = \frac{1}{2} \boldsymbol{\sigma}_\alpha \cdot \mathbf{C} \cdot \boldsymbol{\sigma}_\beta = \frac{\boldsymbol{\sigma}_\alpha \cdot \boldsymbol{\sigma}_\beta}{2 \lambda_\beta} = 0 \quad (17)$$

We say that any two stress and strain states which do not perform work one on another are *energetically independent*.

Definition of a scalar product can be formulated in an arbitrary way as long as all axioms of the scalar product hold true – it maybe any symmetric, positive definite and non-degenerate bilinear form. It is known that for any quadratic form there exists a corresponding bilinear form [44]. If no

locked strain states [43] [45] corresponding with zero eigenvalue of the compliance tensor are taken into consideration, one can see that as the elasticity tensors are symmetric and positive definite all those above mentioned axioms are fulfilled by an energetic scalar product defined as

$$\boldsymbol{\alpha} \bullet \boldsymbol{\beta} \overset{def}{=} \boldsymbol{\alpha} \cdot \mathbf{C} \cdot \boldsymbol{\beta} \qquad \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{T}_{sym}^2 \qquad (18)$$

Any two states which are orthogonal with respect to the energetic scalar product defined above are called energetically orthogonal

$$\boldsymbol{\alpha} \perp \boldsymbol{\beta} \iff \boldsymbol{\alpha} \bullet \boldsymbol{\beta} = 0. \qquad (19)$$

It means that

$$2\Phi(\boldsymbol{\alpha} + \boldsymbol{\beta}) = (\boldsymbol{\alpha} + \boldsymbol{\beta}) \cdot \mathbf{C} \cdot (\boldsymbol{\alpha} + \boldsymbol{\beta}) = \boldsymbol{\alpha} \cdot \mathbf{C} \cdot \boldsymbol{\alpha} + \boldsymbol{\beta} \cdot \mathbf{C} \cdot \boldsymbol{\beta} + 2\boldsymbol{\alpha} \cdot \mathbf{C} \cdot \boldsymbol{\beta}. \qquad (20)$$

If the condition (19) is satisfied, then

$$2\boldsymbol{\alpha} \cdot \mathbf{C} \cdot \boldsymbol{\beta} = 0$$

and the states $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are energetically independent.

It is also worth noting that eigenstate and eigenvalue problem for certain operator mapping linear space into itself can be defined in various ways depending on the assumed definition of the scalar product in that space. If the energetic scalar product (18) is assumed, then an energetic eigenproblem for any linear operator $\mathbf{L}: \mathcal{T}_{sym}^2 \to \mathcal{T}_{sym}^2$ can be formulated. Any stress state $\mathbf{g}$ satisfying, [13]

$$\mathbf{L} \bullet \mathbf{g} \equiv (\mathbf{L} \circ \mathbf{C}) \cdot \mathbf{g} = \frac{1}{2\gamma}\mathbf{g} \iff L_{ijkl}C_{klmn}g_{mn} = \frac{1}{2\gamma}g_{ij} \quad (21)$$

is called an energetic eigenstate of $\mathbf{L}$ for an appropriate energetic eigenvalue $\frac{1}{2\gamma}$. If $\mathbf{L} = \mathbf{S}$, then

$$\mathbf{S} \bullet \mathbf{g} = (\mathbf{S} \circ \mathbf{C}) \cdot \mathbf{g} = \mathbf{I}^S \cdot \mathbf{g} = \mathbf{g} \qquad (22)$$

so the stiffness tensor acts as an identity operator if the energetic scalar product is used to define the inner product of tensors. Since quadratic form

$$\mathbf{f} \bullet \mathbf{L} \bullet \mathbf{g} = \mathbf{f} \cdot (\mathbf{C} \circ \mathbf{L} \circ \mathbf{C}) \cdot \mathbf{g} = f_{ij}C_{ijkl}L_{klmn}C_{mnpq}g_{pq} \quad (23)$$

is symmetric, all of the energetic eigenvalues are real.

2.1.3. Main decomposition of elastic energy density

Any two eigenstates of the elasticity tensors corresponding to different eigenvalues are both orthogonal and energetically orthogonal. It enables rewriting the total elastic energy density

$$\Phi(\boldsymbol{\sigma}) = \frac{1}{2}\boldsymbol{\sigma} \cdot \mathbf{C} \cdot \boldsymbol{\sigma} \qquad (24)$$

as an additive function of its stress or strain argument, what is in general impossible for quadratic forms such as energy:

$$\Phi(\boldsymbol{\sigma}) = \Phi(\boldsymbol{\sigma}_1) + \Phi(\boldsymbol{\sigma}_2) + \ldots + \Phi(\boldsymbol{\sigma}_\rho), \rho \leqslant 6$$
$$\Phi(\boldsymbol{\sigma}_\alpha) = \frac{|\boldsymbol{\sigma}_\alpha|^2}{2\lambda_\alpha}, \qquad \alpha = 1, 2, \ldots, \rho \qquad (25)$$

where $\boldsymbol{\sigma}_\alpha$ (α =1, 2, ..., ρ) are the eigenstates of the compliance tensor. We will call the decomposition (25) the main decomposition of elastic energy density.

2.2. Energy-based hypotheses of material effort for anisotropic solids

2.2.1. Generalized limit condition of Mises and its energetic interpretation – Rychlewski's theorem

Let us considera generalized limit criterion for anisotropic solids by Mises [15] of the following form:

$$\boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} = 1 \qquad (26)$$

where $\mathbf{H}$ is a fourth rank limit state tensor satisfying the internal symmetry conditions (2). In the original formulation of the limit condition of Mises [15], he considered the case when only the deviator of the stress state contribute to the measure of material effort. It is worth noting that Mises emphasized that the limit condition (26) has no energetic sense. $\mathbf{H}$ describes the strength properties of the considered body and it is represented by a matrix of coefficients of a quadratic form of the limit condition (26). If the flow rule associated with the limit condition (26) is considered, then $\mathbf{H}$ is a linear operator in $\mathcal{T}_{sym}^2$ and an eigenproblem can be formulated for it, namely finding such stress states $\mathbf{h}$ and scalars h that, [34]

$$\mathbf{H} \cdot \mathbf{h} = \frac{1}{k^2}\mathbf{h}. \qquad (27)$$

Classical theorem on the spectral decomposition of any linear operator, which was presented in the previous section referring to the elasticity tensors, leads to the following result:

$$\mathbf{H} = \frac{1}{k_1^2}\mathbf{R}_1 + \frac{1}{k_2^2}\mathbf{R}_2 + \ldots + \frac{1}{k_\kappa^2}\mathbf{R}_\kappa, \qquad \kappa \leqslant 6 \qquad (28)$$

where $\frac{1}{k_\alpha^2}$ and $\mathbf{R}_\alpha$ (α =1, 2, ..., κ) are the eigenvalues and orthogonal projectors on the corresponding eigensubspaces of $\mathbf{H}$ respectively. The spectral decomposition maybe also rewritten in the following form,

$$H = \frac{1}{k_I^2}(h_I \otimes h_I) + \frac{1}{k_{II}^2}(h_{II} \otimes h_{II}) + \ldots + \frac{1}{k_{VI}^2}(h_{VI} \otimes h_{VI}), \quad (29)$$

which is unique only for $\frac{1}{k_\alpha^2}$ being single roots of the characteristic polynomial. The limit condition (26) maybe rewritten as:

$$\boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} = \frac{|\boldsymbol{\sigma}_1|^2}{k_1^2} + \frac{|\boldsymbol{\sigma}_2|^2}{k_2^2} + \ldots + \frac{|\boldsymbol{\sigma}_\kappa|^2}{k_\kappa^2} = 1, \qquad \boldsymbol{\sigma}_\alpha = \mathbf{R}_\alpha \cdot \boldsymbol{\sigma}. \qquad (30)$$

If any of the limit stress value $k_\alpha = \infty (\alpha = 1, 2, ..., \kappa)$ then the corresponding stress state $\boldsymbol{\sigma}_\alpha$ maybe considered as a safe stress state as it does not influence the limit condition.

It is important to note here, that both the compliance tensor $\mathbf{C}$ and limit state tensor $\mathbf{H}$ each maybe used as a linear operator in $\mathcal{T}_{sym}^2$ $\mathbf{C}$ in the generalized Hooke's law (1), $\mathbf{H}$ in the associated flow rule – as well as a quadratic form – $\mathbf{C}$ in the expression for elastic energy density (24), $\mathbf{H}$ in the limit condition (26). In the face of this simple fact the following theorem emerges [44].

Theorem

Let $\mathcal{E}$ be an n-dimensional affine space and let there are two quadratic forms defined in it $\mathbf{x} \cdot \mathbf{A} \cdot \mathbf{x}$ and $\mathbf{x} \cdot \mathbf{B} \cdot \mathbf{x}$, $\mathbf{x} \in \mathcal{E}$ and let the form $\mathbf{x} \cdot \mathbf{B} \cdot \mathbf{x}$ be symmetric and positive definite. Then there exists such a basis in which both quadratic form scan be represented in the canonical form.

As it was already mentioned, for any quadratic form $\mathbf{x} \cdot \mathbf{B} \cdot \mathbf{x}$ there exists a bilinear form $\mathbf{x} \cdot \mathbf{B} \cdot \mathbf{y}$ which is polar to it. If it is assumed to be positive definite then the scalar product defined as

$$\langle \mathbf{x} | \mathbf{y} \rangle \overset{def}{=} \mathbf{x} \cdot \mathbf{B} \cdot \mathbf{y} \tag{31}$$

fulfills all axioms of the scalar product. In $\mathcal{E}$ there exists thus a basis which is orthonormal with respect to the scalar product defined above:

$$e_K \cdot \mathbf{B} \cdot e_L = \delta_{KL} \tag{32}$$

in which both quadratic forms $\mathbf{x} \cdot \mathbf{A} \cdot \mathbf{x}$ and $\mathbf{x} \cdot \mathbf{B} \cdot \mathbf{x}$ are simultaneously represented in their canonical form (as a sum of squares).

Rychlewski [13] has used the theorem on the simultaneous reduction of two quadratic forms $\sigma \cdot \mathbf{C} \cdot \sigma$ and $\sigma \cdot \mathbf{H} \cdot \sigma$ to their canonical forms with assumption of the energetic scalar product (18).

Rychlewski's theorem

For every elastic material defined by its compliance tensor $\mathbf{C}$ and limit state tensor $\mathbf{H}$, there exist exactly one energetically orthogonal decomposition of the linear space of symmetric second order tensors $\mathcal{J}_{sym}^2$:

$$\begin{aligned} \mathcal{L}_{sym}^2 &= \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_\chi, \quad \chi \leqslant 6, \\ \mathcal{H}_\alpha &\perp \mathcal{H}_\beta \text{ for } \alpha \neq \beta \quad \alpha, \beta = 1, 2, \ldots, \chi \end{aligned} \tag{33}$$

and exactly one set of pairwise unequal constants

$$h_1, \ldots, h_\chi, \quad h_\alpha \neq h_\beta \text{ for } \alpha \neq \beta \tag{34}$$

such that, for an arbitrary stress state σ

$$\sigma = \sigma_1 + \ldots + \sigma_\chi, \qquad \sigma_\alpha \in \mathcal{H}_\alpha \tag{35}$$

the measure of material effort given by formula (26) is equal

$$\sigma \mathbf{H} \sigma = \frac{1}{h_1} \Phi(\sigma_1) + \ldots + \frac{1}{h_\chi} \Phi(\sigma_\chi), \tag{36}$$

where

$$\Phi(\sigma) = \frac{1}{2} \sigma \cdot \mathbf{C} \cdot \sigma = \Phi(\sigma_1) + \ldots + \Phi(\sigma_\chi) \tag{37}$$

the total elastic energy density.

In the general case the tensors $\mathbf{C}$ and $\mathbf{H}$ are independent. The elasticity tensors $\mathbf{C}$ and the limit state tensor $\mathbf{H}$ are coaxial when they have the same eigensubspaces and thus the same orthogonal projectors, yet in general they may still have di?erent eigenvalues.

If the tensors $\mathbf{C}$ and $\mathbf{H}$ are coaxial then the elastic energy density decomposition (37) is called by Rychlewski [13] *the main energy decomposition* (25). It is important to note, that the decomposition of $\mathcal{J}_{sym}^2$ into the eigensubspaces of the elasticity tensors is the only decomposition which is both orthogonal and energetically orthogonal.

2.2.2. Limit state orthogonality

Rychlewski has used the quadratic form $\sigma \cdot \mathbf{C} \cdot \sigma$ to define the energetic scalar product the form of the Mises limit condition could be chosen as well. According to the postulates of Mises [15] an addition of a hydrostatic component should not influence the limit condition (26) – it is due to assumption of that limit stress is pressure insensitive. It can be shown, that the pressure in sensitivity is equivalent to the statement that any hydrostatic stress state is an eigenstate of $\mathbf{H}$ corresponding with the zero eigenvalue. However, positive definite limit state tensor maybe used in definition of limit state scalar product and limit state orthogonality

$$\alpha \blacktriangle \beta = \alpha \cdot \mathbf{H} \cdot \beta \qquad \alpha, \beta \in \mathcal{J}_{sym}^2 \tag{38}$$

Any two tensors which are orthogonal with respect to the scalar product defined above will be called limit orthogonal

$$\alpha \hat{\perp} \beta \quad \Leftrightarrow \quad \alpha \blacktriangle \beta = 0 \tag{39}$$

According to the theorem in [44] on parallel reduction of two quadratic form sin to the sum of squares, we might analogically represent the total elastic energy density in the following form:

$$\Phi(\sigma) = \frac{1}{2} \sigma \cdot \mathbf{C} \cdot \sigma = \frac{\mathcal{W}(\sigma_1)}{w_1} + \frac{\mathcal{W}(\sigma_2)}{w_2} + \ldots + \frac{\mathcal{W}(\sigma_\vartheta)}{w_\vartheta}, \quad \vartheta \leqslant 6 \tag{40}$$

such that for any stress state σ

$$\begin{aligned} \sigma \cdot \mathbf{H} \cdot \sigma &= W(\sigma_1) + W(\sigma_2) + \ldots + W(\sigma_\chi) \\ W(\sigma_\alpha) &= \sigma_\alpha \cdot \mathbf{H} \cdot \sigma_\alpha = \alpha \blacktriangle \alpha \qquad \alpha, \beta = 1, 2, \ldots, \vartheta \\ \alpha &\neq \beta \Rightarrow \sigma_\alpha \hat{\perp} \sigma_\beta \end{aligned} \tag{41}$$

Limit state orthogonality is yet still difficult in physical interpretation, however it may be a subject of further research.

2.2.3. Elastic energy decompositions – Rychlewski's hypothesis of material effort

Energetically orthogonal decompositions of elastic energy density introduced by Rychlewski [13] became a basis for the formulation of a new proposition of an energy-based limit condition for anisotropic media. Rychlewski proposed to consider the linear combination of the terms of energy decomposition (36) as a measure of material effort. The material is considered to be in the limit state if the energetic measure reaches certain fixed value which is a constant material parameter independent of the applied load.

$$\frac{\Phi_1}{h_1} + \frac{\Phi_2}{h_2} + \ldots + \frac{\Phi_\rho}{h_\rho} = 1, \qquad \rho \leqslant 6 \tag{42}$$

This is energetic interpretation of the limit criterion of Mises type given by Rychlewski [13]. If the considered decomposition of energy density is the main decomposition of elastic

energy density, then the limit condition maybe written in the following form

$$\frac{|\boldsymbol{\sigma}_1|^2}{2\lambda_1 h_1^2} + \frac{|\boldsymbol{\sigma}_2|^2}{2\lambda_2 h_2^2} + \ldots + \frac{|\boldsymbol{\sigma}_\rho|^2}{2\lambda_\rho h_\rho^2} = 1 \qquad (43)$$

or simply

$$\frac{|\boldsymbol{\sigma}_1|^2}{k_1^2} + \frac{|\boldsymbol{\sigma}_2|^2}{k_2^2} + \ldots + \frac{|\boldsymbol{\sigma}_\rho|^2}{k_\rho^2} = 1, \quad k_\alpha^2 = 2\lambda_\alpha h_\alpha^2, \quad (\alpha = 1, \ldots, \rho) \qquad (44)$$

where $\boldsymbol{\sigma}_\alpha$ are the projections of the current stress state σ on the eigensubspaces of the elasticity tensors, λ_α is the corresponding Kelvin modulus and $k_\alpha (\alpha = 1, 2, \ldots, \rho)$ is the limit value of the proper stress that should be determined experimentally for properly choosen modes of stress pertinent to each eigenstate, cf. [49]. For detailed analysis of the limit criteria of the form (44) see [35].

Rychlewski's limit criterion due to its energetic formulation seems to be much better physically motivated than the majority of other, purely empiric propositions. An important feature of this proposition is that these are the elastic properties of a body themselves (Kelvin moduli, elastic proper states) which define the mathematical form of the limit criterion and not an arbitrary choice of an author. This hypothesis lacks however certain desired features. First of all, as it is strictly quadratic function of the stress state (thus symmetric – sign insensitive), it cannot describe materials exhibiting the strength differential effect. Another thing is that the limit values of the energy densities h_α (and in consequence the limit values of the corresponding stresses k_α) are constant. It is in the contrary with the well known fact, that for certain materials, even isotropic ones, the limit shear stress may be different for various shear modes corresponding with different values of the Lode angle.

2.2.4. The concept of the stress state dependent influence parameters – Burzyński's hypothesis of material effort

As it was mentioned above, strictly quadratic energy-based limit criteria cannot account for the asymmetry of the elastic range. This was one of the reasons for which Burzyński modified the well known limit criterion of Huber [10] (who was his teacher) so that it could account for the strength differential effect. He considered only a special class of linear elastic materials for which the decomposition of elastic energy into a sum of energy of volume change Φ_v and energy of distortion Φ_f is possible:

$$\Phi = \underbrace{\frac{1}{2}\mathbf{A}_\sigma \cdot \mathbf{A}_\varepsilon}_{\Phi_v} + \underbrace{\frac{1}{2}\mathbf{D}_\sigma \cdot \mathbf{D}_\varepsilon}_{\Phi_f} \qquad (45)$$

where $\mathbf{A}_\sigma$, $\mathbf{A}_\varepsilon$ and $\mathbf{D}_\sigma$, $\mathbf{D}_\varepsilon$ denote the spherical and deviatoric part of stress and strain tensors respectively. This kind of material we call *volumetrically isotropic*. Such decomposition is

possible when the components of the elasticity tensors fulfill the following – so called –

$$(3\ independent\ relations) \begin{cases} C_{1123} + C_{2223} + C_{3323} = 0 \\ C_{1131} + C_{2231} + C_{3331} = 0 \\ C_{1112} + C_{2212} + C_{3312} = 0 \end{cases}$$
$$(2\ independent\ relations) \begin{cases} C_{1111} - C_{2222} = C_{2233} - C_{1133} \\ C_{2222} - C_{3333} = C_{3311} - C_{2211} \\ C_{3333} - C_{1111} = C_{1122} - C_{3322} \end{cases}$$
$$(46)$$

Burzyński proposed to consider as a measure of material effort a linear combination of the energy density of volume change and energy density of distortion, yet the influence of the pressure (dilatancy) is determined by a stress state dependent parameter η. The formulation of the hypothesis of material effort can be presented as follows:

$$\eta\Phi_v + \Phi_f = K \qquad (47)$$

where K is the limit value of the stored energy density. Due to a special character of the scaled term in the decomposition of energy Burzyński assumed that the parameter η is only pressure sensitive. According to the experimental motivation the following form of this pressure influence function was suggested:

$$\eta(p) = \omega + \frac{\delta}{3p} \qquad (48)$$

where $p = \frac{1}{3}\text{tr}(\sigma)$ is the hydrostatic stress and ω and δ are certain constant material parameters. The Burzyński hypothesis remains still one of the most general propositions for isotropic solids – it anticipated in a particular case widely used criterion of Drucker and Prager [46] and maybe applied for both ductile and brittle materials. The character of the yield surface which, maybe any quadric surface, is uniquely determined by the relations between the limit values of stress at tension, compression and shear. Burzyński made also an at tempt for accounting for anisotropy, yet it has been not completed to provide a theoretically satisfactory criterion – for details see [30]. It is also worthy to note that the problem of volumetric-distortional decomposition considered by Burzyński in 1928 [14] was undertaken recently by Ting [47] and Federico [48].

3. New proposition of a hypothesis of material effort

3.1. General formulation

The authors' new proposition is based generally on two concepts – the idea of energy decompositions and the idea of the stress dependent parameters influencing the measure of material effort. The idea of authors is to extend the hypothesis of material effort of Rychlewski so that it accounted for the materials with asymmetric elastic range in the same way in which Burzyński modified the hypothesis of Huber.

Proposition

Let us consider any energetically orthogonal decomposition of the linear space of symmetric second order tensors:

$$\mathcal{J}_{sym}^2 = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_\mu, \quad \mu \leqslant 6$$
$$\alpha \neq \beta \Leftrightarrow \mathcal{H}_\alpha \perp \mathcal{H}_\beta, \quad \alpha, \beta = 1, 2, \ldots, \mu. \tag{49}$$

We propose that the material reaches the limit state, if the following relation between the parts of elastic energy density holds:

$$\eta_1 \Phi_1 + \eta_2 \Phi_2 + \ldots + \eta_\mu \Phi_\mu = 1, \quad \Phi_\alpha = \frac{1}{2} \sigma_\alpha \cdot \mathbf{C} \cdot \sigma_\alpha, \quad \sigma_\alpha \in \mathcal{H}_\alpha, \tag{50}$$

*where $\eta_\alpha (\alpha = 1, ..., \mu)$ are certain stress state dependent material parameters – we will call them, depending on their physical interpretation, either **influence functions** or **stress mode indicators**.*

The energetically orthogonal decomposition (49) of $\mathcal{J}_{sym}^2$ maybe uniquely determined by certain quadratic form $\sigma \cdot \mathbf{H} \cdot \sigma$ (36) or simply by a single tensor $\mathbf{H}$ – an example of finding such a decomposition with use of a tensor $\mathbf{H}$ of assumed symmetry can be found in [49]. It is important however to emphasize that the new proposition differs from the generalized limit criterion of Mises both in its mathematical formulation as well as in its physical interpretation.

3.2. Assumptions on the stress mode indicators and influence functions

Following assumptions are made on the form and character of the introduced stress mode indicators and in?uence functions:

- **Limit value**
  If we consider a stress state $\sigma = \sigma_\alpha (\alpha = 1, ..., \mu)$ which belongs entirely only to a single subspace $\mathcal{H}_\alpha$ of the decomposition (49), then if the considered stress state reaches the limit state, then the limit condition (50) takes form

  $$\eta_\alpha \left(\sigma_\alpha^{\lim} \right) = \Phi \left(\sigma_\alpha^{\lim} \right)^{-1}.$$

  So the value of the appropriate parameter η_α in the limit state is equal to the inversion of the limit value of the energy density corresponding with the stress states belonging to proper subspace. Let us remind that η_α is not constant – it is a function of both material and stress state, what distinguishes the new proposition form the Rychlewski's condition (42).

- **Domain**
  It seems natural that to keep mutual energetic independence of the considered energy density terms and their correlation with the scaling parameters η_α, it is necessary to assume that those parameters depend only on the stress state belonging to the corresponding subspace $\mathcal{H}_\alpha$. According to (49) any stress state σ maybe decomposed into a sum

  $$\sigma = \sigma_1 + \sigma_2 + \ldots + \sigma_\mu, \quad \sigma_\alpha \in \mathcal{H}_\alpha.$$

  Then each of the parameters η_α is assumed to be a function of an argument being a projection of the current stress state on the corresponding subspace $\mathcal{H}_\alpha$

  $$\eta_\alpha = \eta_\alpha (\sigma_\alpha).$$

The authors suggest, that the parameter η_α should be called the "influence function" if it depends on the norm of its argument – this is when the magnitude of the appropriate stress influences the total measure of material effort in a way which is not proportional to the corresponding energy density. It concerns especially the subspaces of non-deviatoric stress states which may contribute to the strength differential effect. If the considered subspace is multidimensional, in which the elements belonging to it are not just proportional one to another but also their form may vary, the respective parameter η_α will be called the "stress mode indicator". In general, parameter η_α may be both influence function and stress mode indicator.

- **Arguments of η_α**
  Each of the functions $\eta_\alpha (\alpha = 1, ..., \mu)$ is a scalar function of a tensor argument – in general, such functions may have quite complex structure. In any practical calculation such a function may be expressed e.g. in terms of the invariants of σ_α or in terms of its components in a fixed basis. We state that in general case each function η_α may be assumed to be of the following form:

  $$\eta_\alpha = \eta_\alpha (|\sigma_\alpha|, \varphi_1, \varphi_2, \ldots, \varphi_N), \quad N = \dim \mathcal{H}_\alpha - 1,$$

  where non-dimensional parameters $\varphi_k (k = 1, ..., N)$ determine the decomposition of the stress state being an argument of η_α in the set of basis states in $N+1$-dimensional subspace $\mathcal{H}_\alpha$. An analogous quantity commonly used in case of isotropy is the Lode angle. In case of one-dimensional subspace $\mathcal{H}_\alpha$, the measure of the projection of the stress state on that subspace is a complete information on the σ_α if only the basis state in this space is known. It is thus the only needed argument of η_α:

  $$\mathcal{H}_\alpha = lin \{\omega\}, \quad \dim \mathcal{H}_\alpha = 1 \Rightarrow \eta_\alpha = \eta_\alpha (\sigma_\alpha),$$
  $$\sigma_\alpha = \sigma \cdot \omega.$$

  There are many cases in which it seems necessary to consider the functions η_α as anisotropic functions, it means as functions which features change depending e.g. on the decomposition of their argument state in the basis of its multidimensional domain sub-space. However sometimes it may be justified to assume that η_α are isotropic. Their isotropy must not be mistaken with isotropy in physical space – parameters η_α might be isotropic only in their domain, namely in an abstract subspace of an energetically orthogonal decomposition of $\mathcal{J}_{sym}^2$. As the isotropic functions they are in particular expressed solely in terms of the invariants of their arguments (e.g. trace, norm, second or third invariant of the argument or its deviator etc.):

  $$\eta_\alpha = \eta_\alpha (I_1 (\sigma_\alpha), I_2 (\sigma_\alpha), I_3 (\sigma_\alpha))$$

- **Devaiatoric subspaces $\mathcal{H}_\alpha$**
  In case when the SD effect is not observed at shearing, for $\mathcal{H}_\alpha$ being and appropriate subspace of deviators (pure shears or their compositions) we assume on the respective η_α that it is an even function (symmetric, sign insensitive):

  $$\eta_\alpha (-\sigma_\alpha) = \eta_\alpha (\sigma_\alpha).$$

It is worth to mention that if $\boldsymbol{\sigma}_\alpha$ is a deviator, then obviously $I_1(\boldsymbol{\sigma}_\alpha) = 0$ and $I_2(\boldsymbol{\sigma}_\alpha)$ is proportional to $\boldsymbol{\sigma}_\alpha$ – one can see that it is in fact the third invariant of the stress tensor deviator which makes the qualitative (not only quantitative) distinction between various forms of shearing. It is known for certain materials that different forms of shearing cause plastic yield at different level of the shear stress magnitude – it is the well known dependency on the Lode angle, which is strictly connected with the third invariant of the stress tensor deviator. For this reason, the parameters η in case of deviatoric subspaces might be interpreted as the *shear mode indicators*.

It may be assumed for simplicity of the yield criterion formulation that the measure of material effort respective for the state belonging to the deviatoric subspace is proportional to the energy connected with it for any fixed form of such state and thus the corresponding parameter η_α is independent of its norm – it is then the shear mode indicator, however it is not an influence function. We will make such an assumption as well as the one that for any state belonging to any deviatoric subspace of the considered decomposition no SD effect occurs. Furthermore, if the deviatoric subspace $\mathcal{H}_\alpha$ is one-dimensional, then the parameter η_α is proportional to the inverse of the square of the limit value of the considered stress state:

$$\eta_\alpha \sim \frac{1}{k_\alpha^2}$$

The preliminary analysis of the presented hypothesis was presented in [50], [51], however in the current paper more comprehensive exposition of the concepts of influence functions and stress mode indicators is given.

4. Specification of the limit criterion for chosen elastic symmetries

There are two main tasks in the process of specification of a limit criterion for certain material – the choice of the proper energetically orthogonal decomposition of the space of stress and strain tensors and finding the forms of the influence functions. The decomposition of $\mathcal{J}_{sym}^2$ into eigensubspaces of the elasticity tensors seems to be the most natural choice both from physical and mathematical point of view. It can be easily interpreted physically and it is the only such decomposition which is both energetically orthogonal and orthogonal in the sense of a classically defined scalar product. Most of the following specifications are derived with use of the decomposition into eigensubspaces of $\mathbf{C}$ and $\mathbf{S}$, however an example of distinct decomposition is also given. For detailed analysis and specification of the new yield criterion in case of plane stress state see [51].

Let us note that in such a case the assumption of isotropy of the in?uence functions corresponds with the almost forgotten proposition of Mises [15] of a yield function of the stress state dependent scalar arguments which are invariant with respect to all geometric transformations respective for the considered elastic symmetry. The clue difference is that the form of those invariants is not chosen arbitrary as it was done in [15], yet it is uniquely given by the definition of three invariants of the second rank tensors and by the decomposition of $\mathcal{J}_{sym}^2$ into eigensubspaces of the elasticity tensors (e.g.by form of orthogonal projectors).

In the following specifications of the limit criterion for some chosen elastic symmetries the follwing notation for the fourth rank tensors satisfying (2) is used:

$$A = \begin{bmatrix} A_{1111} & A_{1122} & A_{1133} & \sqrt{2}A_{1123} & \sqrt{2}A_{1131} & \sqrt{2}A_{1112} \\ & A_{2222} & A_{2233} & \sqrt{2}A_{2223} & \sqrt{2}A_{2231} & \sqrt{2}A_{2212} \\ & & A_{3333} & \sqrt{2}A_{3323} & \sqrt{2}A_{3331} & \sqrt{2}A_{3312} \\ & & & 2A_{2323} & 2A_{2331} & 2A_{2312} \\ & sym & & & 2A_{3131} & 2A_{3112} \\ & & & & & 2A_{1212} \end{bmatrix} \tag{51}$$

In the following considerations it is assumed that the axes of the coordinate system in physical space coincide with the axes of symmetry or directions normal to the planes of symmetry of the considered anisotropic material.

The spectral decomposition of the elasticity tensors of very low elastic symmetry (triclinic, monoclinic, orthotropic) depends strongly on the numerical values of their components – even the number of orthogonal eigensubspaces may vary in large extent. Kelvin moduli as well as the form of the eigenstates depend in a strongly non-linear way on the components of the elasticity tensors. Those symmetries and also trigonal symmetry are not discussed in the current paper only due to complexity of the notation and great generality of those symmetries which makes the problem of precise specification of the limit criterion for those symmetries rather impractical. For the spectral analysis of the elasticity tensors for all elastic symmetries see e.g. [12], [52], [53].

4.1. Isotropy – main decomposition of elastic energy density

Author's proposition in case of isotropy can be considered as a consistent extension of general energy-based approach of Burzyński accounting for the influence of the Lode angle, which was not involved in the original proposition of Burzyński. In the case of isotropy there are only two eigensubspaces of the elasticity tensors – one-dimensional subspace of spherical tensors and a five-dimensional subspace of deviators. Due to isotropy of the material the problem may be simplified by introducing the principal stresses. The limit condition (50) in case of isotropy may be rewritten in the following form

$$\eta_v(p) \cdot p^2 + \eta_f(\theta) \cdot q^2 = 1 \tag{52}$$

or even in the most general form

$$\eta_p(I_1) + \eta_q(J_2, J_3) - C = 0 \tag{53}$$

where:

$p = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ – hydrostatic sress

$q = \sqrt{\frac{1}{3}\left[(\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + (\sigma_1 - \sigma_2)^2\right]}$ – deviatoric stress

$\theta = \frac{1}{3}\arccos\left[\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{(3/2)}}\right]$ – Lode angle

C – constant material parameter

The isotropic case of the considered limit condition was already discussed [54], [55] and specified for Inconel-718 alloy [56] basing on the experimental data available in the literature [57]. The influence function $\eta_p(p)$ describes the pressure sensitivity of the material, while the shear mode indicator $\eta_q(J_2, J_3)$ might be considered as a function describing the Lode angle dependence $\eta_f(J_2, J_3) = \eta_f(\theta)$. Due to isotropy it is assumed that any function of the Lode angle should be periodic with its period equal 120° – Lode angle dependent functions are of ten considered as a functions of argument $y = \cos 3\theta$. There are many propositions of the pressure in?uence functions or Lode angle dependent shear mode indicators in the literature:

Pressure influence functions
- Two-parameter rational function by Burzyński [14]
  $\eta_p(p) = \left(\omega + \frac{\delta}{p}\right) p^2$
- Five-parameter function by Bigoni and Piccolroaz [58]
  $$\eta_p(p) = \begin{cases} -\frac{2K}{p^2} \cdot M p_c \sqrt{(F - F^m)[2(1 - \alpha)F + \alpha]} \\ \qquad\qquad \text{if} \quad F = \frac{-p + c}{p_c + c} \in [0, 1] \\ +\infty \qquad\quad \text{if} \quad F = \frac{-p + c}{p_c + c} \notin [0, 1] \end{cases}$$

Shear mode indicators
- One-parameter trigonometric function by Lexcellent et. al. [59] $\eta_f(y) = \cos\left[\frac{1}{3}\arccos[1 - \alpha(1 - y)]\right]$
- Two-parameter exponential function by Raniecki and Mróz [60] $\eta_f(y) = 1 + \alpha\left[1 - e^{-\beta(1+y)}\right]$
- Two-parameter power function by Raniecki and Mróz [60] $\eta_f(y) = [1 + \alpha y]^\beta$
- Two-parameter trigonometric function by Podgórski [61] $\eta_f(y) = \frac{1}{\cos(30° - \beta)} \cos\left[\frac{1}{3}\arc\cos(\alpha \cdot y) - \beta\right]$
- Valuable summary of the Lode angle dependent functions can be also found in [62]

In particular the newly introduced limit criterion for isotropic solids (53) can be represented as a generalization of the most of the commonly used limit criteria -e.g. Maxwell-Huber-Mises, Coulomb-Tresca-Guest, Coulomb-Mohr, Drucker-Prager and others. It is enough to assume the Burzyński's pressure influence function with $\omega = 0$ and Podgórski's shear mode indicator multiplied by $\frac{q}{\sqrt{3}}$ and to take the values of the parameters of the Podgórski's functions as indicated in [61] to obtain required results.

4.2. Cubic symmetry – main decomposition of elastic energy density

Cubic symmetry – respective for the regular crystal system (typical for e.g. iron, copper, etc.) – is characterized by three mutually orthogonal four-fold symmetry axes and planes of symmetry perpendicular to those axes as well as by a three-fold symmetry axes which are equally inclined to the symmetry axes mentioned first. The spectral decomposition of a fourth rank symmetric tensor of cubic symmetry gives us a three mutually orthogonal eigensubspaces: one-dimensional eigensubspace $\mathscr{P}_1$ of hydrostatic stress states, two-dimensional eigensubspace $\mathscr{P}_2$ of deviators (compositions of pure-shears in planes of symmetry in directions equally inclined to any two principal symmetry axes) and three-dimensional eigensubspace $\mathscr{P}_3$ of deviators (com-

positions of pure-shears in planes of symmetry in directions of the principal symmetry axes).

The limit condition (50) for cubic symmetry maybe rewritten the following general form:

$$\eta_v(p) \cdot p^2 + \eta_{f1}(\theta) \cdot q_1^2 + \eta_{f2}(\varphi, \psi) \cdot q_2^2 = 1 \qquad (54)$$

where

$p = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$ – hydrostatic stress

$q_1 = \sqrt{\frac{1}{3}\left[(\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + (\sigma_{11} - \sigma_{22})^2\right]}$ – oblique deviatoric stress

$q_2 = \sqrt{\sigma_{23}^2 + \sigma_{31}^2 + \sigma_{12}^2}$ – oblong deviatoric stress

and the non-dimensional quantities θ as well as φ and ψ are parameters determining the form of the projections of the stress state in the two-dimensional and three-dimensional eigensubspace of deviators respectively – for details see [52]. Those quantities fulfill the following relations:

$$\theta = \frac{1}{3}\arccos\left[\frac{3\sqrt{3}}{2}\frac{J_3(\sigma_2)}{J_2^{(3/2)}(\sigma_2)}\right]$$

$$\sin 2\psi \cdot \left[\cos\varphi\left(1 - \cos^2\varphi\right)\right] = \frac{J_3(\sigma_3)}{J_2^{(3/2)}(\sigma_3)}$$

where J_2 and J_3 denote the second and the third invariant of the stress tensor deviator, and $\sigma_K(K = 1, 2)$ denote the projections of the general stress state σ on the corresponding K^{th} eigensubspace $\mathscr{P}_K$. The problem of determination of the pressure influence function η_v and two shear mode indicators η_{f1} and η_{f2} in cubic symmetry is highly complicated due to fact that the stress state at most of typical laboratory tests (uniaxial tests and pure shears in various directions) belongs to at least two eigensubspaces and thus all three functions must be determined simultaneously. However for certain orientations of the applied uniaxial load, the influence of one of the eigenstates may be set to zero. If the uniaxial stress of magnitude k is applied in direction of the elementary cell diagonal [1, 1, 1] then the stress state is orthogonal to the second eigensubspace:

$$p = \frac{k}{3}, \qquad q_1 = 0, \qquad q_2 = \frac{k}{\sqrt{3}}$$

It is also sure that at pure shear in the planes of symmetry, in directions of the symmetry axes one must obtain

$$\eta_{f2}(0°, 0°) = \eta_{f2}(90°, 90°) = \eta_{f2}(90°, 0°) = \frac{1}{k_s^2}$$

where k_s is the limit shear stress for this orientation (it concerns also any other equivalent pairs φ, ψ). Then, in case of uniaxial stress state in which the stress vector lays in one of the principal plane of symmetry, the influence of the third eigenstate is known, and only two unknown functions contribute to the total measure of material effort.

4.3. Volumetrically isotropic orthotropy – main decomposition of elastic energy density

Let us consider a volumetrically isotropic orthotropic compliance tensor – it is characterized by three mutually orthogonal two-fold symmetry axes. The decomposition of the elastic energy density into volumetric part and distortional

part is additionally possible for such a symmetry due to assumption of its volumetrical isotropy. The spectral decomposition of the above compliance tensor gives us six mutually orthogonal eigensubspaces: one-dimensional eigensubspace of hydrostatic stress states, two one-dimensional eigensubspaces of deviatoric stress states (in general they are not pure shears) and three one-dimensional eigensubspaces of pure shears. Let us note that all deviatoric eigensubspaces are one-dimensional – there is no qualitative distinction between any two states belonging to such a subspace. Assuming that the influence functions respective for those subspaces are constant parameters and expressing the energy densities with use of stress state components the general limit condition (50) can be rewritten in the following form:

$$\eta_v(p) \cdot p^2 + \frac{\sigma_{d1}^2}{k_{d1}^2} + \frac{\sigma_{d1}^2}{k_{d1}^2} + \frac{\tau_1^2}{k_{s1}^2} + \frac{\tau_2^2}{k_{s2}^2} + \frac{\tau_3^2}{k_{s3}^2} = 1 \qquad (55)$$

where:

$p = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$ – hydrostatic stress

$\sigma_{d1} = \frac{1}{\sqrt{2(1+\kappa_1+\kappa_1^2)}}(\sigma_{11} + \kappa_1\sigma_{22} - (1+\kappa_1)\sigma_{33})$ – first orthotropic deviatoric stress

$\sigma_{d2} = \frac{1}{\sqrt{2(1+\kappa_2+\kappa_2^2)}}(\sigma_{11} + \kappa_2\sigma_{22} - (1+\kappa_2)\sigma_{33})$ – second orthotropic deviatoric stress

$\tau_1 = \sigma_{23}$ – magnitude of pure shear in x_2x_3 directions

$\tau_2 = \sigma_{31}$ – magnitude of pure shear in x_3x_1 directions

$\tau_3 = \sigma_{12}$ – magnitude of pure shear in x_1x_2 directions

$k_{d1}, k_{d2}, k_{s1}, k_{s2}, k_{s3}$ – limit values of the magnitude of the appropriate shear stress states.

and

$$\kappa_1 = \frac{C_{3311}-C_{2233}-\sqrt{(C_{1122}^2+C_{3311}^2+C_{2233}^2)-(C_{3311}C_{2233}+C_{2233}C_{1122}+C_{1122}C_{3311})}}{C_{1122}-C_{3311}}$$

$$\kappa_2 = \frac{C_{3311}-C_{2233}+\sqrt{(C_{1122}^2+C_{3311}^2+C_{2233}^2)-(C_{3311}C_{2233}+C_{2233}C_{1122}+C_{1122}C_{3311})}}{C_{1122}-C_{3311}}$$

are functions of the stiffness distributor determining the form of the eigenstates. Due to orthogonality of the corresponding eigensubspaces the following equality holds

$$\kappa_1 + \kappa_2 + 2(1 + \kappa_1\kappa_2) = 0.$$

The form of the pressure influence function is to be determined. Some possible forms of such function are given in a summary of the influence function for isotropic solids. Nevertheless the form of this function might be predicted basing on the experimental data – since the term corresponding with the first eigensubspace is the only one which takes into account the hydrostatic stress component and it depends only on its magnitude the form of the pressure influence function can be determined basing on the uniaxial states (which has a non-zero hydrostatic component) in various directions. Uniaxial stress state of magnitude k in direction givenin the considered basis by avers or **n** is represented by a stress tensor

$$\sigma = k(\mathbf{n} \otimes \mathbf{n}) = k\begin{bmatrix} n_1^2 & n_1n_2 & n_1n_3 \\ & n_2^2 & n_2n_3 \\ sym & & n_3^2 \end{bmatrix}$$

where n_i denotes the i-th component of **n** in the considered basis. If the stress reaches the limit state at its limit magnitude

$k_\mathbf{n}$ characteristic for each direction, then the limit condition (55) can be rewritten in the following form:

$$\eta_v\left(\frac{k_n}{3}\right) = 9\left[\frac{1}{k_n} - \frac{(n_1^2 + \kappa_1 n_2^2 - (1+\kappa_1)n_3^2)^2}{2(1+\kappa_1+\kappa_1^2)k_{d1}^2} - \frac{(n_1^2 + \kappa_2 n_2^2 - (1+\kappa_2)n_3^2)^2}{2(1+\kappa_2+\kappa_2^2)k_{d2}^2} - \frac{n_2^2 n_3^2}{k_{s1}^2} - \frac{n_3^2 n_1^2}{k_{s2}^2} - \frac{n_1^2 n_2^2}{k_{s3}^2}\right] \qquad (56)$$

which determines a value of the pressure influence function for certain value of its argument. Sufficiently large set of experimental data enable estimation of the form of $\eta_v(p)$ through appropriate interpolation or approximation.

4.4. Cubic symmetry – decomposition of elastic energy density given by orthotropic tensor H

Let us note that in all of the above examples of the yield criterion specification it was the main decomposition of elastic energy weighted terms of which were considered as a measure of material effort. It is important to emphasize that it is only a specific case of the presented proposition of the hypothesis of material effort. In general any decomposition of the elastic energy density which is energetically orthogonal can be taken into account. According to the Rychlewski's theorem any such decomposition is uniquely related to two quadratic forms, $\sigma \cdot \mathbf{C} \cdot \sigma$ and $\sigma \cdot \mathbf{H} \cdot \sigma$, given by the compliance tensor $\mathbf{C}$ and tensor $\mathbf{H}$ which – in case of the generalized limit condition of Mises – may be interpreted as a limit state tensor. In general, for certain elastic material given by its compliance tensor $\mathbf{C}$ the tensor $\mathbf{H}$ exhibiting certain desired symmetry properties and the quadratic form respective for it might be used in order to find a proper decomposition of the elastic energy density, which is energetically orthogonal and thus may be used in the formulation of the yield criterion according to the proposed hypothesis of material effort. In [49] an energy-based limit condition for cubic materials exhibiting volumetrically isotropic orthotropy of its strength properties in the limit state was considered. In such a case the two deviatoric eigensubspaces of the elasticity tensors may be splitted into five one-dimensional deviatoric subspaces.

$$\mathscr{P}_1 = \mathscr{H}_1, \quad \mathscr{P}_2 = \mathscr{H}_2 \oplus \mathscr{H}_3, \quad \mathscr{P}_3 = \mathscr{H}_4 \oplus \mathscr{H}_5 \oplus \mathscr{H}_6. \quad (57)$$

Numbering of the eigensubspaces $\mathscr{P}$ of cubic compliance tensor $\mathbf{C}$ is the same as in the subsection dedicated to the cubic symmetry. The elastic energy density maybe still decomposed into the terms corresponding with the energetically orthogonal subspaces $\mathscr{H}_\alpha(\alpha = 1, ..., 6)$ and the limit state condition may be written in the form:

$$\eta_v\Phi_v + \eta_{f2}\Phi_{f2} + \eta_{f3}\Phi_{f3} + \eta_{f4}\Phi_{f4} + \eta_{f5}\Phi_{f5} + \eta_{f6}\Phi_{f6} = 1 \quad (58)$$

which in turn maybe rewritten in terms of the components of the stress state, so that it took the final form analogical to (55). The above decomposition is not the main decomposition of the elastic energy density, however due to relations (57) the following equalities hold:

$$\Phi_1 = \Phi_v$$
$$\Phi_2 = \Phi_{f1} + \Phi_{f2}$$
$$\Phi_3 = \Phi_{f3} + \Phi_{f4} + \Phi_{f5}$$

where $\Phi_\alpha(\alpha = 1, 2, 3)$ are the terms of the main decomposition of the elastic energy density corresponding with the proper eigensubspaces of the cubic elasticity tensors $\mathscr{P}_\alpha$.

5. Summary

New proposition of a hypothesis of material effort was presented. Limit criteria for most of elastic spatial elastic symmetries were derived from the proposed hypothesis. Methodology of acquiring the data necessary for the criterion specification was proposed. Methods of prediction of the form of unknown influence functions and stress mode indicators basing on basic strength tests (uniaxial test, pure shear) was proposed. It seems that numerical simulation of the deformation respective for the considered eigenstate (and its mode) of a molecular model exhibiting proper symmetry may enable more precise prediction of the form of those functions assuming that the stored energy (which finds its equivalent on the molecular level) is the measure of material effort. Further research on the proposed hypothesis requires broad and complex experimental verification.

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REFERENCES

[1] T. F r ą ś, Z. N o w a k, P. P e r z y n a, R.B. P ę c h e r s k i, Identification of the model describing viscoplastic behaviour of high strength metals, Inverse Problems in Science and Engineering **19**, 1, 17-30 (2011).

[2] R.B. P ę c h e r s k i, Relation of microscopic observations to constitutive modelling of advanced deformations and fracture initiation of viscoplastic materials, Archives of Mechanics **35**, 257-277 (1983).

[3] J.J. G i l m a n, Electronic Basis of the Strength of Materials, Cambridge University Press, Cambride, UK, 2003.

[4] K. N a l e p k a, R.B. P ę c h e r s k i, Modelling of the interatomic interactions in the Copper crystal applied in the structure (111)Cu-(0001) Al$_2$O$_3$, Archives of Metallurgy and Materials **54**, 511-522 (2009).

[5] K.T. N a l e p k a, R.B. P ę c h e r s k i, The Strength of the interfacial bond in the ceramic matrix composites Al$_2$O$_3$-Ni, Mechanics and Control **29**, 132-137 (2010).

[6] K. N a l e p k a, Symmetry-based approach to parametrization of embedded-atom-method interatomicpotentials, Computational Materials Science **56**, 100-107 (2012).

[7] K. N a l e p k a, Efficient approach to metal/metal oxide interfaces within variable charge model, The European Physical Journal B **85**, 45 (2012).

[8] J.C. M a x w e l l, Origins of Clerk Maxwell's electric ideas as described in familiar letters to William Thomson, Cambridge University Press, 1937.

[9] E. B e l t r a m i, Opere matem. **4**, Mailand, 180-189 (1920).

[10] M.T. H u b e r, Właściwa praca odkształcenia jako miara wytężenia materiału. Czasopismo Techniczne, 15, Lwów, 1904; see also: Specific work of strain as a measure of material effort, Archives of Mechanics **56**, 3, 173-190 (2004).

[11] W. T h o m s o n, (Lord Kelvin) Elements of a mathematical theory of elasticity, Philosophical Transactions of the Royal Society **166**, 481-498 (1856).

[12] J. R y c h l e w s k i, On Hooke's law [in Russian]. Prikladnaya Matematika i Mekhanika, 48, 420-435 (1984); see also: On Hooke's law. Journal of Applied Mathematics and Mechanics **48**, 303-314 (1984).

[13] J. R y c h l e w s k i, Elastic energy decomposition and limit criteria [in Russian]. Uspekhi mekhaniki, 7, 51-80 (1984); see also: Elastic energy decomposition and limit criteria. Engineering Transactions **59**, 1, 31-63 (2011).

[14] W. B u r z y ń s k i, Studium nad hipotezami wytężenia, Akademia Nauk Technicznych, Lwów, 1928; see also: Selected passages from Włodzimierz Burzyński's doctoral dissertation Study on material effort hypotheses. Engineering Transactions **57**, 3-4, 185-215 (2009).

[15] R. M i s e s, Mechanik der plastischen Formanderung von Kristallen. Zeitschrift für Angewandte Mathematik und Mechanik **8**, 161-185 (1928).

[16] R. M i s e s, Mechanik der festen Körper im plastisch deformablen Zustand. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse **1**, 582-592 (1913).

[17] R. H i l l, A theory of the yielding and plastic flow of anisotropic metals. Proceedings of the Royal Society of London **193**, 281-297 (1948).

[18] O. H o f f m a n, The brittle strength of orthotropic materials. Journal of Composite Materials **1**, 200-206 (1967).

[19] S.W. T s a i, E.M. W u, A general theory of strength for anisotropic materials. Journal of Composite Materials **5**, 58-80 (1971).

[20] R.M. C a d d e l l, R.S. R a g h a v a, A.G. A t k i n s, Ayield criterion for anisotropic and pressure dependent solids such as orientedpolymers. Journal of Materials Science **8**, 11, 1641-1646 (1973).

[21] V.S. D e s h p a n d e, N.A. F l e c k, M.F. A s h b y, Effective properties of theoctet-truss lattice material. Journal of the Mechanics and Physics of Solids **49**, 1747-1769 (2001).

[22] C. L i u, Y. H u a n g, M.G. S t o u t, On the asymmetric yield surface of plastically orthotropic materials; a phenomenological study. Acta Materialia **45**, 6, 2397-2406 (1997).

[23] R. H i l l, Theoretical plasticity of textured aggregates. Mathematical Proceedings of the Cambridge Philosophical Society **85**, 1, 179-191 (1979).

[24] W.F. H o s f o r d, On yield loci of anisotropic cubic metals. Proceedings of the 7th North American Metalworking Conference, SME, Dearborn, MI., (1979).

[25] D. B a n a b i c, Sheet metal forming processes, constitutive modelling and numerical simulation, Chapter 2, Springer Verlag, Berlin, Heidelberg, 2010.

[26] L.W. H u, Modified Tresca's yield condition and associated flow rules for anisotropic materials and applications. Journal of the Franklin Institute **265**, 3, 187-204 (1958).

[27] A.P. K a r a f i l l i s, M.C. B o y c e, A general anisotropic yield criterion using bounds and a transformation weighting tensor. Journal of the Mechanics and Physics of Solids **41**, 1859-1886 (1993).

[28] H.L. S c h r e y e r, Q.H. Z u o, Anisotropic yield surfaces based on elastic projection operators. Journal of Applied Mechanics **62**, 780-785 (1995).

[29] J.M. L u q u e - R a i g ó n, R. C a m p o a m o r - S t u r s - b e r g, A unified approach for plasticity yield criteria on the tangent space to the Cauchy tensor. Mathematics and Mechanics of Solids **17**, 2, 83-103 (2012).

[30] P. Szeptyński, Some remarks on Burzyński's failure criterion for anisotropic materials. Engineering Transactions **59**, 2, 119-136 (2011).

[31] W. Olszak, W. Urbanowski, The plastic potential and the generalized distortion energy in the theory of non-homogeneous anisotropic elastic-plastic bodies. Archives of Mechanics **8**, 671-694 (1956).

[32] I.I. Goldenblat, Theory of small elastic-plastic deformations of anisotropic media [in Russian], Izvestiya Akademii Nauk SSSR, Otd.Tehn. Nauk, 2, (1955).

[33] I.I. Goldenblat, Some problems of the mechanics of deformable media [in Russian], Gostekhizdat, Moscow, 1955.

[34] W. Olszak, J. Ostrowska-Maciejewska, The plastic potential in the theory of anisotropic elastic-plastic solids. Engineering Fracture Mechanics **21**, 4, 625-632 (1985).

[35] J. Ostrowska-Maciejewska, J. Rychlewski, Plane elastic and limit states in anisotropic solids. Archives of Mechanics **40**, 4, 379-386 (1988).

[36] W.A. Spitzig, R.J. Sober, O. Richmond, The Effect of hydrostatic pressure on the deformation behavior of maraging and HY-80 steels and its implications for plasticity theory. Metallurgical Transactions A **7A**, 1704-1710 (1976).

[37] O. Richmond, W.A. Spitzig, Pressure dependence and dilatancy of plastic flow. Theoretical and applied mechanics; Proceedings of the 15th International Congress, 377-386 (1980).

[38] C.D. Wilson, A critical reexamination of classical metal plasticity. Journal of Applied Mechanics **69**, 63-68 (2002).

[39] D.C. Drucker, Plasticity theory, strength-differential (SD) phenomenon, and volume expansion in metals and plastics. Metallurgical Transactions **4**, 667-673 (1973).

[40] O. Cazacu, F. Barlat, Generalization of Drucker's yield criterion to orthotropy. Mathematics and Mechanics of Solids **6**, 6, 613-630 (2001).

[41] M.M. Mehrabadi, S.C. Cowin, Eigentensors of linear anisotropic elastic materials, Quarterly Journal of Mechanics and Applied Mathematics **43**, 1, 15-41 (1990).

[42] S. Sutcliffe, Spectral decomposition of the elasticity tensor, Transactions of the ASME 762, **59** (1992).

[43] J. Rychlewski, Unconventional approach to linear elasticity, Archives of Mechanics **47**, 2, 149-171 (1995).

[44] I.M. Gelfand, Lectures on linear algebra [in Russian], Nauka, Moscow, 1966.

[45] K. Kowalczyk-Gajewska, J. Ostrowska-Maciejewska, The influence of internal restrictions on the elastic properties of anisotropic materials, Archives of Mechanics **56**, 3, 205-232 (2004).

[46] D.C. Drucker, W. Prager, Soil mechanics and plastic analysis for limit design. Quarterly of Applied Mathematics **10**, 2, 157-165 (1952).

[47] T.C.T. Ting, Can a linear anisotropic elastic material have a uniform contraction under a uniform pressure? Mathematics and Mechanics of Solids **6**, 235-243 (2001).

[48] S. Federico, Volumetric-distortional decomposition of deformation and elasticity tensor. Mathematics and Mechanics of Solids **15**, 672-690 (2010).

[49] K. Kowalczyk, J. Ostrowska-Maciejewska, R.B. Pęcherski, An energy-based yield criterion for solids of cubic elasticity and orthotropic limit state. Archives of Mechanics **55**, 5-6, 431-448 (2003).

[50] J. Ostrowska-Maciejewska, R.B. Pęcherski, P. Szeptyński, Limit condition for anisotropic materials with asymmetric elastic range, Engineering Transactions **60**, 2, 125-138 (2012).

[51] P. Szeptyński, R.B. Pęcherski, Proposition of a new yield criterion for orthotropic metal sheets accounting for asymmetry of elastic range [in Polish]. Rudy i Metale Nieżelazne **57**, 4, 243-250 (2012).

[52] K. Kowalczyk-Gajewska, J. Ostrowska-Maciejewska, Review on spectral decomposition of Hooke's tensor for all symmetry groups of linear elastic material. Engineering Transactions **57**, 3-4, 145-183 (2009).

[53] A. Blinowski, J. Ostrowska-Maciejewska, On the elastic orthotropy, Archives of Mechanics **48**, 1, 129-141 (1996).

[54] M. Nowak, J. Ostrowska-Maciejewska, R.B. Pęcherski, P. Szeptyński, Yield criterion accounting for the third invariant of stress tensor deviator. Part I. Derivation of the yield condition basing on the concepts of energy-based hypotheses of Rychlewski and Burzyński. Engineering Transactions **59**, 4, 273-281 (2011).

[55] P. Szeptyński, Yield criterion accounting for the influence of the third invariant of stress tensor deviator. Part II: Analysis of convexity condition of the yield surface. Engineering Transactions **59**, 4, 283-297 (2011).

[56] R.B. Pęcherski, P. Szeptyński, M. Nowak, An extension of Burzyński hypothesis of material effort accounting for the third invariant of stress tensor, Archives of Metallurgy and Materials **56**, 2, 503-508 (2011).

[57] S.K. Iyer, C.J. Lissenden, Multiaxial constitutive model accounting for the strength-differential in Inconel 718. International Journal of Plasticity **19**, 2055-2081 (2003).

[58] D. Bigoni, A. Piccolroaz, Yield criteria for quasibrittle and frictional materials. International Journal of Solids and Structures **41**, 2855-2878 (2004).

[59] C. Lexcellent et al., Experimental and numerical determinations of the initial surface of phase transformation under biaxial loadingin some polycrystalline shape-memory alloys. Journal of the Mechanics and Physics of Solids **50**, 2717-2735 (2002).

[60] B. Raniecki, Z. Mróz, Yield or martensitic phase transformation conditions and dissipation functions for isotropic, pressure-insensitive alloys exhibiting SD effect. Acta Mechanica **195**, 81-102 (2008).

[61] J. Podgórski, Limit state condition and the dissipation function for isotropic materials. Archives of Mechanics **36**, 3, 323-342 (1984).

[62] J.P. Bardet, Lode dependences for isotropic pressure sensitive materials. Journal of Applied Mechanics **57**, 498-506 (1990).