Noise Analysis of Differential Multiport Networks – the Wave Approach

Janusz A. Dobrowolski

Abstract—**In this paper there is presented and discussed a general analysis method for noise characterization of noisy multiport differential networks. It is based on mixed mode, differential and common mode, noise waves representation of noise, generalized mixed-mode scattering parameters and generalized mixed-mode noise wave correlation parameters for the network. There are derived analytical relation between the noise figure for a given output port and the noise matrix and the scattering parameters of the network, as well as the correlations between the input port noise waves. The signal to noise ratio degradation factor is derived and discussed, too. Presented results can be implemented directly in a CAD software noise analysis of differential microwave multiport networks with differential as well as with conventional single ended ports.**

Keywords—**Differential networks, noise analysis, noise waves, noise correlation matrix, differential noise figure**

I. INTRODUCTION

MANY present day RF and microwave networks are implemented as differential networks. In comparison to standard single ended networks, differential are more resistible against external disturbances and noise, particularly supply and ground noise. Differential networks require special tools for characterization, analysis and design. D. Bockelman and W.R. Eisenstadt {1] have introduced so-called mixed-mode waves (wave variables) and mixed mode scattering parameters to extend the classical single-ended wave approach to the differential case. In 2006, A. Ferrero and M. Pirolla [2] introduced generalized mixed-mode scattering matrix which can be used for hybrid networks having some ports differential and some ports single-ended. Such theory may be used for characterization and signal analysis and design of differential networks containing differential amplifiers, baluns, transformers etc.

The noise wave formalism has been applied for years to study noise wave properties of single ended two-port devices and networks [3-4] and multi-port networks [5-6]. J. Randa [7] has proposed a way to apply single–ended noise waves and standard single-ended scattering parameters to derive noise parameters of differential multiport networks. In [8-10] mixed mode noise wave formalism and mixed mode S parameters are applied to derive the differential noise figure of four port differential networks.

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In this paper there is presented a general noise analysis method which is based on mixed mode, differential- and common-mode, noise wave formalism [12] derived from pseudo-wave definition presented in [11] by R. Marks and D. Williams, the generalized mixed-mode scattering matrix [2] and the mixed mode noise wave correlation matrix [12]. Presented approach may be used for noise analysis of differential networks with mixed mode ports as well as with single-ended ports. The method is applied to evaluate the differential noise figure and the differential signal to noise ratio degradation factor of mixed mode multiport networks.

II. NOISE ANALYSIS OF MIXED MODE MULTIPORT NETWORKS

This approach to the analysis of noisy microwave multiport mixed-mode networks is based on assumption that a linear noisy network can be represented by the single $n \times n$ generalized mixed-mode scattering matrix $\overset{o}{\mathbf{S}}$ and a vector $\overset{o}{\mathbf{c}}$ of the equivalent noise wave sources representing all intrinsic noise sources in the multiport [1,2,12]. For such network we can write

$$\overset{o}{\mathbf{b}} = \overset{o}{\mathbf{S}}\,\overset{o}{\mathbf{a}} + \overset{o}{\mathbf{c}} \tag{1}$$

where $\overset{o}{\mathbf{a}}$ and $\overset{o}{\mathbf{b}}$ are vectors of the ingoing and outgoing generalized mixed-mode noise wave amplitudes and $\overset{o}{\mathbf{c}}$ is a vector of amplitudes of the generalized mixed-mode equivalent noise wave sources representing the intrinsic noise sources in the multiport network.

One-port terminations of the multiport are characterized by the equation

$$\overset{o}{\mathbf{a}} = \overset{o}{\mathbf{\Gamma}}\,\overset{o}{\mathbf{b}} + \overset{o}{\mathbf{c}}_L \tag{2}$$

in which $\overset{o}{\mathbf{c}}_L$ is the vector of the generalized mixed-mode equivalent noise wave sources representing thermal noise generated in terminations (in loads and in internal impedances of the signal sources) connected to the ports of the multiport, and $\overset{o}{\mathbf{\Gamma}}$ is the $n \times n$ reflection coefficients matrix of signal sources and loads connected to the n ports of the multiport.

Vectors $\overset{o}{\mathbf{a}}$, $\overset{o}{\mathbf{b}}$, $\overset{o}{\mathbf{c}}$ and $\overset{o}{\mathbf{c}}_L$ are the generalized mixed-mode noise wave vectors, which combine the p mixed-mode port set of noise pseudowaves and the remaining $(n - p)$ single-ended port set of noise pseudowaves

$$\overset{o}{\mathbf{a}} = \begin{bmatrix} \mathbf{a}_d \\ \mathbf{a}_c \\ \mathbf{a}_e \end{bmatrix} \qquad \overset{o}{\mathbf{b}} = \begin{bmatrix} \mathbf{b}_d \\ \mathbf{b}_c \\ \mathbf{b}_e \end{bmatrix} \qquad \overset{o}{\mathbf{c}} = \begin{bmatrix} \mathbf{c}_d \\ \mathbf{c}_c \\ \mathbf{c}_e \end{bmatrix} \qquad \overset{o}{\mathbf{c}_L} = \begin{bmatrix} \mathbf{c}_{Ld} \\ \mathbf{c}_{Lc} \\ \mathbf{c}_{Le} \end{bmatrix}$$

(3)

The first $p/2$ elements in vectors $\overset{o}{\mathbf{a}}$, $\overset{o}{\mathbf{b}}$, $\overset{o}{\mathbf{c}}$ and $\overset{o}{\mathbf{c}_L}$ are named, respectively, $\mathbf{a}_d$, $\mathbf{b}_d$, $\mathbf{c}_d$ and $\mathbf{c}_{Ld}$. Elements of these vectors are the differential-mode noise wave variables at the mixed-mode ports of the network. The second set of $p/2$ elements in vectors $\overset{o}{\mathbf{a}}$, $\overset{o}{\mathbf{b}}$, $\overset{o}{\mathbf{c}}$ and $\overset{o}{\mathbf{c}_L}$ are named, respectively, $\mathbf{a}_c$, $\mathbf{b}_c$, $\mathbf{c}_c$ and $\mathbf{c}_{Lc}$. Elements of these vectors are the common-mode noise wave variables at the mixed mode ports of the network. And finally, the last $(n-p)$ elements in vectors $\overset{o}{\mathbf{a}}$, $\overset{o}{\mathbf{b}}$, $\overset{o}{\mathbf{c}}$ and $\overset{o}{\mathbf{c}_L}$ are named, respectively, $\mathbf{a}_e$, $\mathbf{b}_e$ $\mathbf{c}_e$ and $\mathbf{c}_{Le}$. Elements of these vectors are the single-ended noise wave variables at the single-ended ports of a multiport network.

The matrix $\overset{o}{\mathbf{\Gamma}}$ in (2) is the $(n \times n)$ matrix with entries equal to the reflection coefficients seen at each signal source port or load port of the network

$$\overset{o}{\mathbf{\Gamma}} = \begin{bmatrix} \mathbf{\Gamma}_d & \mathbf{\Gamma}_{dc} & \mathbf{0} \\ \mathbf{\Gamma}_{cd} & \mathbf{\Gamma}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Gamma}_e \end{bmatrix}$$

(4)

In (4), the submatrix $\mathbf{\Gamma}_d$ is the $(p/2 \times p/2)$ diagonal matrix with the differential-mode pseudowave reflection coefficients at ports of the differential signal sources and the differential loads on the main diagonal. The submatrix $\mathbf{\Gamma}_d$ is given by (5).

$$\mathbf{\Gamma}_d = \begin{bmatrix} \Gamma_{d12} & 0 & .. & .. & & 0 \\ 0 & \Gamma_{d34} & .. & .. & & : \\ : & & .. & . & .. & : \\ : & & .. & & .. & . & 0 \\ 0 & & .. & & .. & 0 & \Gamma_{d(p-1)p} \end{bmatrix}$$

(5)

The matrix $\mathbf{\Gamma}_c$ in (4) is the $(p/2 \times p/2)$ diagonal matrix with the common-mode pseudowave reflection coefficients at ports of the differential sources and the differential loads on the main diagonal. The matrix $\mathbf{\Gamma}_c$ is given as

$$\mathbf{\Gamma}_c = \begin{bmatrix} \Gamma_{c12} & 0 & .. & .. & & 0 \\ 0 & \Gamma_{c34} & .. & .. & & : \\ : & & .. & . & .. & : \\ : & & .. & & .. & . & 0 \\ 0 & & .. & & .. & 0 & \Gamma_{c(p-1)p} \end{bmatrix}$$

(6)

The matrix $\mathbf{\Gamma}_e$ in (4) is the $(n-p) \times (n-p)$ diagonal matrix with the single-ended mode pseudowave reflection coefficients at ports of the single-ended signal sources and the single-ended loads on the main diagonal. The matrix $\mathbf{\Gamma}_e$ is given by

$$\mathbf{\Gamma}_e = \begin{bmatrix} \Gamma_{e(p+1)} & 0 & .. & .. & 0 \\ 0 & \Gamma_{e(p+2)} & .. & .. & : \\ : & & .. & . & .. & : \\ : & & .. & & .. & . & 0 \\ 0 & & .. & & .. & 0 & \Gamma_{en} \end{bmatrix}$$

(7)

The matrices $\mathbf{\Gamma}_{dc}$ and $\mathbf{\Gamma}_{cd}$ in (4) are also the $(n-p) \times (n-p)$ diagonal matrices. They are given by (8) and (9).

$$\mathbf{\Gamma}_{dc} = \begin{bmatrix} \Gamma_{dc12} & 0 & .. & .. & & 0 \\ 0 & \Gamma_{dc34} & .. & .. & & : \\ : & & .. & . & .. & : \\ : & & .. & & .. & . & 0 \\ 0 & & .. & & .. & 0 & \Gamma_{dc(p-1)p} \end{bmatrix}$$

(8)

$$\mathbf{\Gamma}_{cd} = \begin{bmatrix} \Gamma_{cd12} & 0 & .. & .. & & 0 \\ 0 & \Gamma_{cd34} & .. & .. & & : \\ : & & .. & . & .. & : \\ : & & .. & & .. & . & 0 \\ 0 & & .. & & .. & 0 & \Gamma_{cd(p-1)p} \end{bmatrix}$$

(9)

The main-diagonal elements of the matrices $\mathbf{\Gamma}_{dc}$ and $\mathbf{\Gamma}_{cd}$ are the mode conversion reflection coefficients of particular differential signal sources and of particular differential loads.

Eliminating the vector $\mathbf{a}$ from (1) and (2), we receive the equation for the outgoing noise wave vector versus the waves of noise sources

$$\overset{o}{\mathbf{b}} = \left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\mathbf{\Gamma}} \right]^{-1} \left[\overset{o}{\mathbf{S}}\,\overset{o}{\mathbf{c}_L} + \overset{o}{\mathbf{c}} \right]$$

(10)

in which $\mathbf{I}$ is the identity matrix.

The noise wave correlation matrix for the outgoing noise pseudowaves at all ports of the multiport is

$$\mathbf{N} = \overline{\overset{o}{\mathbf{b}}\,\overset{o}{\mathbf{b}}{}^{+}}$$

(11)

The diagonal elements of the noise correlation matrix $\mathbf{N}$ are equal to the power spectral densities of the output noise in respective ports of the multiport. The off-diagonal elements of $\mathbf{N}$ are the correlation between the output noise in different ports of the network. Using (10) and (11), we can write

$$\mathbf{N} = \overline{\overset{o}{\mathbf{b}}\overset{o}{\mathbf{b}}^{+}} = \left[\mathbf{I} - \overset{o}{\mathbf{S}}\overset{o}{\Gamma}\right]^{-1}\overset{o}{\mathbf{S}}\overline{\overset{o}{\mathbf{c}}_{\mathrm{L}}\overset{o}{\mathbf{c}}_{\mathrm{L}}^{+}}\overset{o}{\mathbf{S}}^{+}\left[\mathbf{I} - \overset{o}{\mathbf{S}}\overset{o}{\Gamma}\right]^{-1+}$$
$$+ \left[\mathbf{I} - \overset{o}{\mathbf{S}}\overset{o}{\Gamma}\right]^{-1}\overline{\overset{o}{\mathbf{c}}\overset{o}{\mathbf{c}}^{+}}\left[\mathbf{I} - \overset{o}{\mathbf{S}}\overset{o}{\Gamma}\right]^{-1+}$$
$$= \left[\mathbf{I} - \overset{o}{\mathbf{S}}\overset{o}{\Gamma}\right]^{-1}\overset{o}{\mathbf{S}}\overset{o}{\mathbf{C}}_{\mathrm{L}}\overset{o}{\mathbf{S}}^{+}\left[\mathbf{I} - \overset{o}{\mathbf{S}}\overset{o}{\Gamma}\right]^{-1+}$$
$$+ \left[\mathbf{I} - \overset{o}{\mathbf{S}}\overset{o}{\Gamma}\right]^{-1}\overset{o}{\mathbf{C}}_{\mathrm{S}}\left[\mathbf{I} - \overset{o}{\mathbf{S}}\overset{o}{\Gamma}\right]^{-1+} \tag{12}$$

In deriving (12) we have taken into account the obvious fact that the noise waves originated in the multiport are not correlated with noise waves generated in the terminations.

The generalized noise wave correlation matrix

$$\overset{o}{\mathbf{C}}_{\mathrm{S}} = \overline{\overset{o}{\mathbf{c}}\overset{o}{\mathbf{c}}^{*}} \tag{13}$$

in (12) is the intrinsic noise matrix of the multiport network, while

$$\overset{o}{\mathbf{C}}_{\mathrm{L}} = \overline{\overset{o}{\mathbf{c}}_{\mathrm{L}}\overset{o}{\mathbf{c}}_{\mathrm{L}}^{+}} \tag{14}$$

s the generalized correlation matrix for the equivalent noise pseudowave sources due to the thermal noise of passive terminations loading each port of the multiport network.

Based on (3) and (9), the matrix $\overset{o}{\mathbf{C}}_{\mathrm{L}}$ may be written as {14]

$$\overset{o}{\mathbf{C}}_{\mathrm{L}} = \begin{bmatrix} \mathbf{C}_{\mathrm{Ld}} & \mathbf{C}_{\mathrm{Ldc}} & \mathbf{0} \\ \mathbf{C}_{\mathrm{Ldc}}^{*} & \mathbf{C}_{\mathrm{Lc}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_{\mathrm{Le}} \end{bmatrix} = kT_0 \overset{o}{\mathbf{A}} = kT_0 \begin{bmatrix} \mathbf{A}_{\mathrm{Ld}} & \mathbf{A}_{\mathrm{Ldc}} & \mathbf{0} \\ \mathbf{A}_{\mathrm{Ldc}}^{*} & \mathbf{A}_{\mathrm{Lc}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{\mathrm{Le}} \end{bmatrix} \tag{15}$$

Because the thermal noise generated in differential and single-ended terminations loading each port of the network is uncorrelated, all noise correlation submatrices presented in (15) are the diagonal matrices. The elements of the matrix $\mathbf{A}_{\mathrm{Ld}}$ corresponding to the differential mode noise waves produced by differential terminations are given by

$$A_{\mathrm{Ld}ii} = \frac{T_{\mathrm{a}i}}{T_0}\left(1 - \left|\Gamma_{\mathrm{d}i}\right|^2 - \left|\Gamma_{\mathrm{dc}i}\right|^2\right) \tag{16}$$

where $\Gamma_{\mathrm{d}i}$ is the differential mode reflection coefficient of the differential signal source port or the differential load port with the port number i, $\Gamma_{\mathrm{dc}i}$ is the mode conversion reflection coefficient of the differential signal source port or of the differential load port with the port number i, and $T_{\mathrm{a}i}$ is the physical absolute temperature of the signal source internal impedance or load impedance.

The elements of the matrix $\mathbf{A}_{\mathrm{Lc}}$ corresponding to the common mode noise waves produced by differential terminations are given by

$$A_{\mathrm{Lc}ii} = \frac{T_{\mathrm{a}i}}{T_0}\left(1 - \left|\Gamma_{\mathrm{c}i}\right|^2 - \left|\Gamma_{\mathrm{cd}i}\right|^2\right) \tag{17}$$

where $\Gamma_{\mathrm{c}i}$ is the common mode reflection coefficient of the differential signal source port or the differential load port with the port number i, $\Gamma_{\mathrm{cd}i}$ is the mode conversion reflection coefficient of the differential signal source port or the differential load port with the port number i, and $T_{\mathrm{a}i}$ is the physical absolute temperature of the signal source internal impedance or load impedance.

And finally, the elements of the matrix $\mathbf{A}_{\mathrm{Ldc}}$ and $\mathbf{A}_{\mathrm{Ldc}}^{*}$, corresponding to the common-mode to differential-mode and differential-mode to common-mode noise waves conversion produced by differential terminations, are given by

$$A_{\mathrm{Ldc}ii} = \frac{T_{\mathrm{a}i}}{T_0}\left(\Gamma_{\mathrm{d}i}\Gamma_{\mathrm{cdi}}^{*} + \Gamma_{\mathrm{ci}}^{*}\Gamma_{\mathrm{cdi}}\right) \tag{18}$$

where, as in (11) and (12), $\Gamma_{\mathrm{d}i}$ is the differential mode reflection coefficient, $\Gamma_{\mathrm{c}i}$ is the common mode reflection coefficient, $\Gamma_{\mathrm{cd}i}$ of the differential signal source port or the differential load port with the port number i, $\Gamma_{\mathrm{dc}i}$ and $\Gamma_{\mathrm{cd}i}$ are the mode conversion reflection coefficients of the differential signal source port or the differential load port with the port number i, and $T_{\mathrm{a}i}$ is the physical absolute temperature of the signal source internal impedance or the load impedance.

The reflection coefficients $\Gamma_{\mathrm{d}i}$, $\Gamma_{\mathrm{c}i}$, $\Gamma_{\mathrm{dc}i}$ and $\Gamma_{\mathrm{cd}i}$ are discussed in Appendix and are given by (D3).

The diagonal elements of the matrix $\overset{o}{\mathbf{C}}_{\mathrm{L}}$ corresponding to the single-ended terminations are given by

$$A_{\mathrm{Le}ii} = \frac{T_{\mathrm{a}i}}{T_0}\left(1 - \left|\Gamma_i\right|^2\right) \tag{19}$$

where Γ_i is the reflection coefficient of the single-ended signal source port or the single-ended load port with the port number i, and $T_{\mathrm{a}i}$ is the physical absolute temperature of the signal source internal impedance or load impedance.

In the case of the complex reference impedance Z_{R} of the termination ports, we have

$$A_{\mathrm{Le}ii} = \frac{T_{\mathrm{a}i}}{T_0}\mathrm{Re}\left[Z_{\mathrm{R}}\frac{\mathrm{Re}(Z_{\mathrm{R}})}{\left|Z_{\mathrm{R}}\right|^2}\left[1 - \left|\Gamma_i\right|^2 + 2j\mathrm{Im}(\Gamma_i)\right]\right] \tag{20}$$

The interpretation of both terms in (12) is very obvious. The first term is related to noise generated in the signal sources and loads connected to the input and output ports of the multiport network, while the second term is related to the noise generated in the network. In the case when all external noise sources are

equal to zero, $\mathbf{C}_L = 0$, (for example, when the temperature of all terminations equals 0 K) and the first term in (12) vanishes. Similarly, a case of noiseless amplifier corresponds to $\mathbf{C}_S = 0$, and the second term in (12) vanishes.

Using (15), the equation (12) may be written in the form

$$\mathbf{N} = \overline{\overset{o}{\mathbf{b}}\,\overset{o}{\mathbf{b}}{}^{+}} = kT_0\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1}\overset{o}{\mathbf{S}}\,\overset{o}{\mathbf{A}}\,\overset{o}{\mathbf{S}}{}^{+}\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1+} + \left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1}\overset{o}{\mathbf{C}}_S\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1+} \tag{21}$$

in which $T_0 = 290$ K is reference noise temperature assumed common to all input ports of the network.

III. DIFFERENTIAL NOISE FIGURE OF MIXED-MODE MULTIPORT NETWORKS

In the definition of two-port network noise figure it is assumed that the termination of the output port is noiseless or in other words that the definition does not include noise generated in the output load [13]. In the definition of the noise figure of the multiport network we use the same convention that there is not noise source located in the load of the output port.

With such definition of the noise figure, we can write the differential noise figure of a multiport network as

$$F = \frac{\left\{\begin{array}{l} kT_0\,\boldsymbol{\delta}_l^{+}\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1}\overset{o}{\mathbf{S}}\,\overset{o}{\mathbf{A}}\,\overset{o}{\mathbf{S}}{}^{+}\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1+}\boldsymbol{\delta}_l \\ + \boldsymbol{\delta}_l^{+}\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1}\overset{o}{\mathbf{C}}_S\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1+}\boldsymbol{\delta}_l \end{array}\right\}}{kT_0\,\boldsymbol{\delta}_l^{+}\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1}\overset{o}{\mathbf{S}}\,\overset{o}{\mathbf{A}}\,\overset{o}{\mathbf{S}}{}^{+}\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1+}\boldsymbol{\delta}_l}$$

$$= 1 + \frac{1}{kT_0}\frac{\boldsymbol{\delta}_l^{+}\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1}\overset{o}{\mathbf{C}}_S\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1+}\boldsymbol{\delta}_l}{\boldsymbol{\delta}_l^{+}\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1}\overset{o}{\mathbf{S}}\,\overset{o}{\mathbf{A}}\,\overset{o}{\mathbf{S}}{}^{+}\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1+}\boldsymbol{\delta}_l} \tag{22}$$

where $\boldsymbol{\delta}_l$ is a vector whose elements are all zeros except a 1 in position l and l is the network output port number.

IV. DIFFERENTIAL SIGNAL TO NOISE DEGRADATION FACTOR OF MIXED-MODE MULTIPORT NETWORKS

In general case the signal-to-noise ratio degradation is not equal to the noise figure.

Assuming that i is the number of the input port of the signal channel and l is the port number of the output port of the multiport, than the output signal power density entering the load at port l is

$$s_{\text{out}} = \boldsymbol{\delta}_l^{+}\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1}\mathbf{A}_S\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1+}\boldsymbol{\delta}_l\;s_{\text{in}} \tag{23}$$

where the matrix

$$\mathbf{A}_S = \begin{array}{c} \\ \\ i \\ \\ \\ \end{array}\begin{array}{c} \quad\quad\quad i \\ \begin{bmatrix} 0 & 0 & 0 & 0 & .. & .. \\ 0 & 0 & 0 & 0 & .. & .. \\ 0 & 0 & 1 & 0 & .. & .. \\ 0 & 0 & 0 & 0 & .. & .. \\ . & . & : & . & . & . \\ . & . & : & . & . & . \end{bmatrix} \end{array} \tag{24}$$

$\boldsymbol{\delta}_l$ is a vector whose elements are all zeros except a 1 in position l and s_{in} is the input signal power density.

Because the power density of noise entering the signal channel input port is

$$n_{in} = \boldsymbol{\delta}_i^{+}\overline{\overset{o}{\mathbf{c}}_L\,\overset{o}{\mathbf{c}}_L{}^{+}}\boldsymbol{\delta}_i = kT_0\boldsymbol{\delta}_i^{+}\overset{o}{\mathbf{A}}\,\boldsymbol{\delta}_i \tag{25}$$

and the total output noise power density in output port l is given by

$$n_{\text{out}} = \boldsymbol{\delta}_l^{+}\left\{\begin{array}{l} kT_0\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1}\overset{o}{\mathbf{S}}\,\overset{o}{\mathbf{A}}\,\overset{o}{\mathbf{S}}{}^{+}\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1+} \\ + \left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1}\overset{o}{\mathbf{C}}_S\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1+} \end{array}\right\}\boldsymbol{\delta}_l \tag{26}$$

where $\overset{o}{\mathbf{A}}$ is the incident noise correlation matrix for the actual network, than the degradation factor of the signal-to-noise ratio is given by

$$\frac{(s/n)_{in}}{(s/n)_{out}} =$$

$$\frac{\boldsymbol{\delta}_l^{+}\left\{\begin{array}{l} kT_0\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1}\overset{o}{\mathbf{S}}\,\overset{o}{\mathbf{A}}\,\overset{o}{\mathbf{S}}{}^{+}\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1+} \\ + \left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1}\overset{o}{\mathbf{C}}_S\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1+} \end{array}\right\}\boldsymbol{\delta}_l}{\left[kT_0\,\boldsymbol{\delta}_i^{+}\overset{o}{\mathbf{A}}\,\boldsymbol{\delta}_i\right]\left\{\boldsymbol{\delta}_l^{T}\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1}\overset{o}{\mathbf{S}}\,\mathbf{A}_S\,\overset{o}{\mathbf{S}}{}^{+}\left[\mathbf{I} - \overset{o}{\mathbf{S}}\,\overset{o}{\Gamma}\right]^{-1+}\boldsymbol{\delta}_l\right\}} \tag{27}$$

When the signal source is matched (27) simplifies to

$$
\frac{(s/n)_{in}}{(s/n)_{out}} = \frac{\boldsymbol{\delta}_l^+ \left\{ \begin{array}{l} kT_0\left[\mathbf{I}-\overset{o}{\mathbf{S}}\overset{o}{\boldsymbol{\Gamma}}\right]^{-1} \overset{o}{\mathbf{S}}\overset{o}{\mathbf{A}}\overset{o+}{\mathbf{S}}\left[\mathbf{I}-\overset{o}{\mathbf{S}}\overset{o}{\boldsymbol{\Gamma}}\right]^{-1+} \\ +\left[\mathbf{I}-\overset{o}{\mathbf{S}}\overset{o}{\boldsymbol{\Gamma}}\right]^{-1}\overset{o}{\mathbf{C}}_S\left[\mathbf{I}-\overset{o}{\mathbf{S}}\overset{o}{\boldsymbol{\Gamma}}\right]^{-1+} \end{array} \right\}\boldsymbol{\delta}_l}{kT_0 \;\; \boldsymbol{\delta}_l^T\left[\mathbf{I}-\overset{o}{\mathbf{S}}\overset{o}{\boldsymbol{\Gamma}}\right]^{-1}\overset{o}{\mathbf{S}}\overset{o}{\mathbf{A}}_S\overset{o+}{\mathbf{S}}\left[\mathbf{I}-\overset{o}{\mathbf{S}}\overset{o}{\boldsymbol{\Gamma}}\right]^{-1+}\boldsymbol{\delta}_l}
\tag{28}
$$

This relation is different to (22) which determines the noise figure of a mixed-mode multiport network because in (28) the total output noise density of the network is divided by the output noise coming from the incident noise in the signal channel only. In (22) determining F, the total output noise is divided by the output noise due to all the incident noise in all input pots of the network.

In the case when all terminations of the network are matched (28) reduces to

$$
\frac{(s/n)_{in}}{(s/n)_{out}} = \frac{\boldsymbol{\delta}_l^+\left[kT_0\overset{o}{\mathbf{S}}\overset{o}{\mathbf{A}}\overset{o}{\mathbf{S}}+\overset{o}{\mathbf{C}}_S\right]\boldsymbol{\delta}_l}{kT_0\;\;\boldsymbol{\delta}_l^+\overset{o}{\mathbf{S}}\overset{o}{\mathbf{A}}_S\overset{o+}{\mathbf{S}}\;\;\boldsymbol{\delta}_l}
\tag{29}
$$

the noise waves.

V. CONCLUSION

The noise analysis concept presented in this paper is applicable to differential multiport networks with mixed mode, differential- and common-mode ports as well as with single-ended ports. Therefore it is applicable to most networks occurring in microwave practice, in particular, to differential amplifiers. The set of parameters which can be calculated by the method includes the differential noise figure and signal to noise degradation factor of multiports with mixed-mode portsa and single-ended ports.

APPENDIX

REFLECTION COEFFICIENT MATRIX OF THE DIFFERENTIAL LOAD

Figure 1 illustrates a passive single-ended two-port network considered as the differential load.
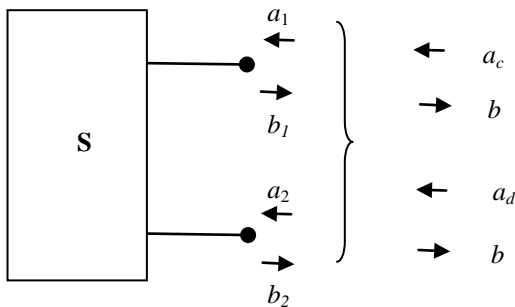

Fig. 1. Passive single-ended two-port considered as a differential load

Matrix equation describing the differential load in the mixed-mode pseudoscattering matrix and mixed-mode pseudowave domain is

$$
\overset{o}{\mathbf{b}} = \overset{o}{\boldsymbol{\Gamma}}_L\; \overset{o}{\mathbf{a}}
\tag{D1}
$$

or

$$
\begin{bmatrix} b_d \\ b_c \end{bmatrix} = \begin{bmatrix} \Gamma_{Ld} & \Gamma_{Ldc} \\ \Gamma_{Lcd} & \Gamma_{Lc} \end{bmatrix}\begin{bmatrix} a_d \\ a_c \end{bmatrix}
\tag{D2}
$$

where $\overset{o}{\mathbf{a}}$ and $\overset{o}{\mathbf{b}}$ are vectors of, respectively, incident and reflected mixed-mode pseudowaves at the port of the differential load, and $\overset{o}{\boldsymbol{\Gamma}}_L$ is the mixed-mode reflection coefficient matrix of the load. In (D2), Γ_{Ld} is the differential-mode pseudowave reflection coefficient, Γ_{Lc} is the common-mode pseudowave reflection coefficient, while Γ_{Ldc} and Γ_{Lcd} are the mode-coupling reflection coefficients.

Assuming now that the reference impedances of the single ended ports of a mixed mode load are the same: are the same: $Z_{R1} = Z_{R2} = Z_R$, and that the differential-mode and the common-mode reference impedances of the mixed-mode signal source ports satisfy relations: $Z_{Rd} = 2\; Z_R$, $Z_{Rc} = Z_R/2$ and in consequence that [2]

$$
a_d = \frac{a_1 - a_2}{\sqrt{2}}
\tag{D3}
$$

$$
b_d = \frac{b_1 - b_2}{\sqrt{2}}
\tag{D\$}
$$

$$
a_c = \frac{a_1 + a_2}{\sqrt{2}}
\tag{D5}
$$

$$
b_c = \frac{b_1 + b_2}{\sqrt{2}}
\tag{D6}
$$

we get equation describing the differential load

$$
\begin{aligned}
\overset{o}{\boldsymbol{\Gamma}}_L &= \begin{bmatrix} \Gamma_{Ld} & \Gamma_{Ldc} \\ \Gamma_{Lcd} & \Gamma_{Lc} \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} S_{11} - S_{12} - S_{21} + S_{22} & S_{11} + S_{12} - S_{21} - S_{22} \\ S_{11} - S_{12} + S_{21} - S_{22} & S_{11} + S_{12} + S_{21} + S_{22} \end{bmatrix}
\end{aligned}
\tag{D7}
$$

In (D7) S_{ij}, $i,j = 1,\, 2$, are standard, single–ended scattering parameters of the differential load.

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