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Fractional descriptor observers for fractional descriptor continuous-time linear system

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Fractional descriptor full-order observers for fractional descriptor continuous-time linear systems are proposed. Necessary and sufficient conditions for the existence of the observers are established. The design procedure of the observers is demonstrated on two numerical examples.

Key words: fractional descriptor linear systems, design, full-order, fractional, observer

1. Introduction

The fractional linear systems have been considered in many papers and books [24,11,14-16]. Positive linear systems consisting of n subsystems with different fractional orders have been proposed in [14, 16]. Descriptor (singular) linear systems have been investigated in [1-9, 13, 18-22, 25, 26]. The eigenvalues and invariants assignment by state and input feedbacks have been addressed in [4,12,18]. The computation of Kronecker's canonical form of a singular pencil has been analyzed in [25].

A new concept of perfect observers for linear continuous-time systems has been proposed in [10]. Observers for fractional linear systems have been addressed in [17,23] and for descriptor linear systems in [6].

In this paper fractional descriptor full-order observers for fractional descriptor continuous-time linear systems will be proposed and necessary and sufficient conditions for the existence of the observer will be established.

The paper is organized as follows. In section 2 the fractional descriptor linear continuous-time systems and their stability are addressed. In section 3 the fractional descriptor observers are introduced and the necessary and sufficient conditions for their existence are established. Concluding remarks are given in section 4.

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2. Fractional descriptor systems and their stability

Consider the fractional descriptor continuous-time linear system

$$E\frac{d^{\alpha}x}{dt^{\alpha}} = Ax + Bu, \quad x_0 = x(0), \tag{1a}$$

$$y = Cx \tag{1b}$$

where $\frac{d^{\alpha}x}{dt^{\alpha}}$ is the fractional α-order derivative defining by Caputo

$${}_0D_t^{\alpha} = \frac{d^{\alpha}x(t)}{dt^{\alpha}} = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{\frac{d^n x}{dt^n}}{(t-\tau)^{\alpha-n+1}}d\tau, \quad n-1 < \alpha < n \in N = \{1,2,...\}, \tag{2}$$

$\Gamma(x) = \int_0^{\infty} e^{-t}t^{x-1}dt$ is the gamma function, $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors, $E, A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$. It is assumed that $\det E = 0$ and

$$\det[E\lambda - A] \neq \text{ for some } \lambda \in \mathcal{C} \text{ (the field of complex number)}. \tag{3}$$

Let U be the set of admissible inputs $u(t) \in U \in \Re^m$ and $X_0 \subset \Re^n$ be the set of consistent initial conditions $x_0 \in X_0$ for which the equation (1) has a solution $x(t)$ for $u(t) \in U$.

Theorem 5 *The solution of the equation* (1) *for* $x_0 \in X_0$ *is given by*

$$x(t) = \sum_{k=-\mu}^{\infty}\left\{\frac{\Psi_k t^{k\alpha}}{\Gamma(k\alpha+1)}x_0 + \int_0^t \frac{\Psi_k(t-\tau)^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}Bu(\tau)d\tau\right\}. \tag{4}$$

Proof Using the Laplace transform and taking into account that [16]

$$\mathcal{L}\left[\frac{d^{\alpha}x}{dt^{\alpha}}\right] = s^{\alpha}X(s) - s^{\alpha-1}x_0 \tag{5}$$

we obtain

$$Es^{\alpha}X(s) - s^{\alpha-1}x_0 = AX(s) + BU(s) \tag{6}$$

where

$$X(s) = \mathcal{L}[x(t)] = \int_0^{\infty} x(t)e^{-st}dt, \quad U(s) = \mathcal{L}[u(t)].$$

From (6) we have

$$X(s) = [Es^{\alpha} - A]^{-1}\left\{s^{\alpha-1}x_0 + BU(s)\right\}. \tag{7}$$

Let

$$[Es^{\alpha}-A]^{-1}=\sum_{k=-\mu}^{\infty}\Psi_k s^{-(k+1)\alpha}. \tag{8}$$

Comparison of the matrices at the same powers of s^{α} of the equality

$$[Es^{\alpha}-A]\left(\sum_{k=-\mu}^{\infty}\Psi_k s^{-(k+1)\alpha}\right)=\left(\sum_{k=-\mu}^{\infty}\Psi_k s^{-(k+1)\alpha}\right)[Es^{\alpha}-A]=I_n \tag{9}$$

yields

$$\begin{bmatrix} E & 0 & 0 & \dots & 0 & 0 & 0\\ -A & E & 0 & \dots & 0 & 0 & 0\\ 0 & -A & E & \dots & 0 & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & -A & E & 0\\ 0 & 0 & 0 & \dots & 0 & -A & E \end{bmatrix}\begin{bmatrix}\Psi_{-\mu}\\ \Psi_{1-\mu}\\ \dots\\ \Psi_{-1}\\ \Psi_0\\ \dots\\ \Psi_q\end{bmatrix}=\begin{bmatrix}0\\0\\ \dots\\ I_n\\ 0\\ \dots\\ 0\end{bmatrix}. \tag{10}$$

It is easy to show that the equation (10) has always a solution $\Psi_{-\mu},\Psi_{1-\mu},\dots,\Psi_q$ for any $q\in N$ if the condition (3) is satisfied [13]. The matrices $\Psi_{-\mu},\Psi_{1-\mu},\dots,\Psi_q$ can be also found by expansion (8) of the matrix $[Es^{\alpha}-A]^{-1}$.

Substituting (8) into (7) we obtain

$$\begin{aligned} X(S) &= \sum_{k=-\mu}^{\infty}\Psi_k s^{-(k+1)\alpha}\left\{s^{\alpha-1}x_0+BU(s)\right\}\\ &= \sum_{k=-\mu}^{\infty}\Psi_k s^{-(k\alpha+1)}x_0+\sum_{k=-\mu}^{\infty}\Psi_k B s^{-(k\alpha+1)}U(s). \end{aligned} \tag{11}$$

Applying to (11) the inverse Laplace transform and the convolution theorem we obtain (4) since $L[t^{\alpha}]=\dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ [16]. □

Definition 1 *The fractional descriptor linear system* (1) *is called asymptotically stable if* $\lim_{t\to\infty}x(t)=0$ *for any finite* $x_0\in X_0$ *and* $u(t)=0$.

Theorem 6 *The fractional descriptor linear system* (1) *is asymptotically stable if the zeros (the eigenvalues of* (E,A)*)* $\lambda_1,\dots,\lambda_p$ *of the equation*

$$\det[E\lambda-A]=\lambda^p+a_{p-1}\lambda^{p-1}+\dots+a_1\lambda+a_0=0 \tag{12}$$

satisfy the condition

$$|\arg\lambda_k|>\alpha\frac{\pi}{2}\quad\text{for}\quad k=1,\dots,p. \tag{13}$$

The eigenvalues satisfying the condition (13) are located in the stability region shown in Fig. 1 and denoted by S_r.

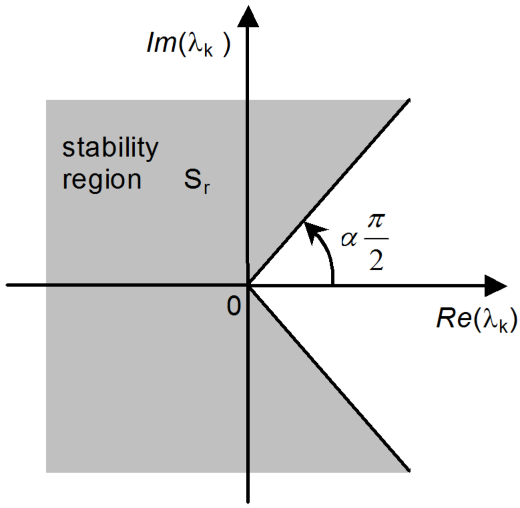

Figure 1. Stability region.

Proof From (4) for $u(t) = 0$, $t \geqslant 0$ we have

$$x(t) = \sum_{k=-\mu}^{\infty} \frac{\Psi_k t^{k\alpha}}{\Gamma(k\alpha+1)} x_0 \tag{14}$$

and if and only if the condition (13) is met then $\lim_{t\to\infty} x(t) = 0$ for any finite $x_0 \in \Re^n$ since [16]

$$\lim_{\substack{t\to\infty \\ k\to\infty}} \frac{\Psi_k t^{k\alpha}}{\Gamma(k\alpha+1)} = 0 \quad \text{and} \quad \lim_{\substack{t\to\infty \\ k\to\infty}} \left[\sum_{k=-\mu}^{\infty} \frac{\Psi_k t^{k\alpha}}{\Gamma(k\alpha+1)} x_0 \right] = 0. \tag{15}$$

Therefore, by definition 1 the fractional descriptor system (1) is asymptotically stable. □

Example 1 Consider the fractional descriptor system (1) with the matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \in \Re^{2\times 2}, \quad B = 0. \tag{16}$$

The condition (3) is satisfied since

$$\det[E\lambda - A] = \det \begin{bmatrix} \lambda - a_1 & 0 \\ 0 & -a_2 \end{bmatrix} = a_2(a_1 - \lambda). \tag{17}$$

In this case the expansion (8) has the form

$$[Es^{\alpha}-A]^{-1}=\begin{bmatrix}\frac{1}{s^{\alpha}-a_1} & 0\\ 0 & -\frac{1}{a_1}\end{bmatrix}=\sum_{k=-\mu}^{\infty}\Psi_k s^{-(k+1)\alpha} \tag{18}$$

where

$$\Psi_k=\begin{cases}\begin{bmatrix}0 & 0\\ 0 & \frac{-1}{a_2}\end{bmatrix} & \text{for} \quad k=-1\\ \\ \begin{bmatrix}a_1^k & 0\\ 0 & 0\end{bmatrix} & \text{for} \quad k=0,1,\dots\end{cases} \tag{19}$$

It is easy to check that the matrices (19) satisfy the equation (4) for $q=3$

$$\begin{bmatrix}E & 0 & 0 & 0 & 0\\ -A & E & 0 & 0 & 0\\ 0 & -A & E & 0 & 0\\ 0 & 0 & -A & E & 0\\ 0 & 0 & 0 & -A & E\end{bmatrix}\begin{bmatrix}\Psi_{-1}\\ \Psi_0\\ \Psi_1\\ \Psi_2\\ \Psi_3\end{bmatrix}=\begin{bmatrix}0\\ I_2\\ 0\\ 0\\ 0\end{bmatrix}. \tag{20}$$

The solution (4) of the equation (1) with (16) has the form

$$\begin{aligned}x(t) &= \sum_{k=-\mu}^{\infty}\frac{\Psi_k t^{k\alpha}}{\Gamma(k\alpha+1)}x_0 \qquad (21)\\ &= \left\{\begin{bmatrix}0 & 0\\ 0 & -a_2^{-1}\end{bmatrix}\frac{t^{-\alpha}}{\Gamma(1-\alpha)}+\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\frac{1}{\Gamma(1)}+\begin{bmatrix}a_1 & 0\\ 0 & 0\end{bmatrix}\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\dots\right\}x_0.\end{aligned}$$

The fractional descriptor system is asymptotically stable for $a_1>0$ and arbitrary a_2.

3. Full-order state observers

Consider the fractional descriptor linear system (1) satisfying the assumption (3).

Definition 2 *The fractional descriptor continuous-time linear system*

$$E\frac{d^{\alpha}\hat{x}}{dt^{\alpha}}=F\hat{x}+Gu+Hy \tag{22}$$

$\hat{x}=\hat{x}(t)\in\Re^n$ *is the estimate of* $x(t)$*, and* $u=u(t)\in\Re^m$, $y=y(t)\in\Re^p$ *are the same input and output vectors as in* (1), $E,F\in\Re^{n\times n}$, $G\in\Re^{n\times m}$, $H\in\Re^{n\times p}$, $\det E=0$ *is called a (full-order) state observer for the system (1) if*

$$\lim_{t\to\infty}[x(t)-\hat{x}(t)]=0. \tag{23}$$

Let

$$e(t) = x(t) - \hat{x}(t). \tag{24}$$

From (24), (1) and (22) we have

$$\begin{aligned} E\frac{d^{\alpha}e(t)}{dt^{\alpha}} &= E\frac{d^{\alpha}x(t)}{dt^{\alpha}} - E\frac{d^{\alpha}\hat{x}(t)}{dt^{\alpha}} = Ax(t) + Bu(t) - (F\hat{x}(t) + Gu(t) + HCx(t)) \\ &= (A - HC)x(t) - F\hat{x}(t) + (B - G)u(t). \end{aligned} \tag{25}$$

If the matrices F, G, H are chosen so that

$$F = A - HC \tag{26a}$$

$$G = B \tag{26b}$$

then from (25) we obtain

$$E\frac{d^{\alpha}e(t)}{dt^{\alpha}} = Fe(t). \tag{27}$$

If the system (27) (or equivalently the pair (E,F)) is asymptotically stable then $\lim\limits_{t\leftarrow\infty} e(t) = 0$ and the observer (2) asymptotically reconstructs the state vector $x(t)$ of the system (1). Therefore, the following theorem has been proved.

Theorem 7 *The fractional descriptor system* (1) *has a full state observer* (22) *if and only if there exists a matrix H such that all eigenvalues of the pair $(E,A-HC)$ are located in the stable region S_c shown on Fig.* 1, *i.e*

$$\sigma(E,A-HC) \subset S_r \tag{28}$$

where σ denotes the spectrum of the pair.

From theorem 7 it follows that the design of a stable observer (22) of the system (1) has been reduced to finding a matrix H such that the eigenvalues of the pair $(E,A-HC)$ are located in the asymptotic stability region. It is well-known [6,13] that there exists a matrix H such that the eigenvalues of the pair $(E,A-HC)$ are located in the asymptotic stability region if and only if the fractional descriptor system (1) is detectable [6,13], i.e.

$$\text{rank}\begin{bmatrix} Es_k - A \\ C \end{bmatrix} = n \quad \text{for} \quad s_k \in \sigma(E,A). \tag{29}$$

The problem of designing of the observer (22) of the system (1) can be reduced to the procedure of designing of a state-feedback $v = -H^T x$ for the dual system [6,13]

$$E^T\frac{d^{\alpha}x}{dt^{\alpha}} = A^T x + C^T v. \tag{30}$$

To guarantee that the descriptor state observer is impulse-free the matrix H must be chosen so that

$$\deg[\det(Ez - A + HC)] = \operatorname{rank} E \tag{31}$$

It is well-known [6,13] that the finite observers poles (the finite eigenvalues of the pair $(E, A - HC)$ can be arbitrary assigned if and only if the descriptor system (1) is R-observable

$$\operatorname{rank}\begin{bmatrix} Es - A \\ C \end{bmatrix} = n \quad \text{for} \quad s \in \mathcal{C} \quad \text{(field of complex umbers)} \tag{32}$$

Therefore, the following theorem has been proved.

Theorem 8 *There exists the impulse-free fractional descriptor observer* (22) *with arbitrary prescribed set of poles of the fractional descriptor system* (1) *satisfying* (3) *if and only if the conditions* (31) *and* (32) *are met.*

Example 2 Consider the fractional system (1) for $0 < \alpha < 1$ with the matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \ (a \neq 0), \quad B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{33}$$

The system satisfies the assumption (3) since

$$[\det(Es - A] = \begin{bmatrix} s & 0 \\ 0 & -a \end{bmatrix} = -as \neq 0. \tag{34}$$

For the system (1) with (33) the conditions (32) and (31) are also satisfied

$$\operatorname{rank}\begin{bmatrix} Es - A \\ C \end{bmatrix} = \operatorname{rank}\begin{bmatrix} s & 0 \\ 0 & -a \\ 1 & 0 \end{bmatrix} = 2 \quad \text{for} \quad a \neq 0 \quad \text{and} \quad s \in \mathcal{C} \tag{35}$$

and for $H = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$ we have

$$\deg[\det(Ez - A + HC)] = \deg\begin{vmatrix} s + h_1 & 0 \\ h_2 & -a \end{vmatrix} = 1 = \operatorname{rank} E = \operatorname{rank}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{36}$$

Therefore, by theorem 8 for the system (1) with (33) there exists the impulse-free fractional descriptor observer (22). Let $s_d < 0$ be the desired pole of the observer. Then using (33) we obtain

$$\det(Es - A + HC)] = \begin{vmatrix} s + h_1 & 0 \\ h_2 & -a \end{vmatrix} = -a(s + h_1) \tag{37}$$

and $s = -h_1 = s_d$, h_2 – arbitrary.

From (26) we have

$$F = A - HC = \begin{bmatrix} s_d & 0 \\ h_2 & a \end{bmatrix}, \quad G = B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{38}$$

The desired observer (22) for the system (1) with (33) has the form

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{d^\alpha \hat{x}}{dt^\alpha} = \begin{bmatrix} s_d & 0 \\ h_2 & a \end{bmatrix} \hat{x} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} u + \begin{bmatrix} -s_d \\ h_2 \end{bmatrix} y. \tag{39}$$

Example 3 Consider the fractional descriptor system (1) for $0 < \alpha < 1$ with the matrices

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{40}$$

The system satisfies the assumption (3) and conditions (32), (31) since

$$\det[Es - A] = \begin{bmatrix} 0 & 0 & -1 \\ 0 & s & 0 \\ s+1 & 0 & 0 \end{bmatrix} = s(s+1) \neq 0 \tag{41}$$

and

$$\operatorname{rank} \begin{bmatrix} Es - A \\ C \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 0 & 0 & -1 \\ 0 & s & 0 \\ s+1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = 3 \quad \text{for all} \quad s \in C \tag{42}$$

and we have

$$\begin{aligned} \deg[\det(Ez - A + HC)] &= \deg \begin{vmatrix} h_{11} & h_{12} & -1 \\ h_{21} & h_{22} + s & 0 \\ h_{31} + 1 + s & h_{32} & 0 \end{vmatrix} = 2 \\ &= \operatorname{rank} E = \operatorname{rank} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{aligned} \tag{43}$$

for suitable choice of the matrix H

$$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ h_{31} & h_{32} \end{bmatrix}.$$

Therefore, by theorem 8 there exists for the system the impulse -free fractional descriptor observer (22).

Let $s_1 = s_2 = -10$ be the desired poles of the observer then (40) we obtain

$$\det(Es - A + HC)] = \begin{vmatrix} h_{11} & h_{12} & -1 \\ h_{21} & h_{22}+s & 0 \\ h_{31}+1+s & h_{32} & 0 \end{vmatrix} = (s+10)^2 \tag{44}$$

for $h_{11} = h_{12} = h_{21} = h_{32} = 0$ and $h_{22} = 10$, $h_{31} = 9$. From (26) and (42) we have

$$F = A - HC = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 10 \\ 9 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -10 & 0 \\ -10 & 0 & 0 \end{bmatrix},$$
$$G = B = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 \\ 0 & 10 \\ 9 & 0 \end{bmatrix}. \tag{45}$$

The desired fractional descriptor observer (22) for the system (1) with (40) has the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \frac{d^\alpha \hat{x}}{dt^\alpha} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -10 & 0 \\ -10 & 0 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 10 \\ 9 & 0 \end{bmatrix} y. \tag{46}$$

4. Concluding remarks

Fractional descriptor full-order observers for fractional descriptor continuous-time linear systems have been proposed. The solution to the equation describing the fractional descriptor continuous-time linear systems has been derived. The necessary and sufficient conditions for the existence of the observer have been established. Designing of the fractional descriptor observers has been illustrated on two numerical examples.

The considerations can be easily extended to the fractional descriptor reduced-order observers and to the fractional observers for fractional descriptor discrete-time linear systems.

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