10.24425/acs.2020.135851

Archives of Control Sciences
Volume 30(LXVI), 2020
No. 4, *pages 757–773*

On one algorithm for reconstruction of an disturbance in a linear system of ordinary differential equations

M. BLIZORUKOVA and V. MAKSIMOV

The problem of reconstructing an unknown disturbance under measuring a part of phase coordinates of a system of linear differential equations is considered. Solving algorithm is designed. The algorithm is based on the combination of ideas from the theory of dynamical inversion and the theory of guaranteed control. The algorithm consists of two blocks: the block of dynamical reconstruction of unmeasured coordinates and the block of dynamical reconstruction of an input.

Key words: dynamical reconstruction, guaranteed control, stable algorithm

1. Introduction. Problem statement

Consider the linear system of differential equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + By(t) + f_1(t), \qquad t \in T = [0, \vartheta], \\ \dot{y}(t) &= Cx(t) + Dy(t) + Eu(t) + f_2(t) \end{aligned} \tag{1}$$

with the initial state

$$x(0) = x_0, \qquad y(0) = y_0 .$$

Here, $0 < \vartheta < +\infty$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^N$, $u \in \mathbb{R}^r$, $f_1(\cdot) \in W^{1,\infty}(T;\mathbb{R}^n) = \{p(\cdot) \in L_2(T;\mathbb{R}^n) : \dot{p}(\cdot) \in L_\infty(T;\mathbb{R}^n)\}$, and $f_2(\cdot) \in L_2(T;\mathbb{R}^N)$ are given functions, u is a disturbance, A, B, C, D, and E are matrices of corresponding dimensions. The problem under consideration consists in the following. Some unknown disturbance $u(\cdot) \in L_2(T;\mathbb{R}^r)$ acts on system (1). At discrete, frequent enough, times

$$\tau_i \in \Delta = \{\tau_i\}_{i=0}^m \quad (\tau_0 = 0, \quad \tau_{i+1} = \tau_i + \delta, \quad \tau_m = \vartheta)$$

Authors are with Krasovskii Institute of Mathematics and Mechanics of Ural Branch of the Russian Academy of Sciences, Yekaterinburg, 620990 Russia. Corresponding author: V. Maksimov, Email: maksimov@imm.uran.ru

The work is done within the framework of research of Ural Mathematical Center.
Received 5.08.2020.

a part of coordinates of the phase state $x(\tau_i) = x(\tau_i; z_0, u(\cdot))$ of system (1) is measured. Here and below, $z_0 = \{x_0, y_0\}$, $z(\cdot; z_0, u(\cdot)) = \{x(\cdot; z_0, u(\cdot)), y(\cdot; z_0, u(\cdot))\}$ is the solution of system (1) corresponding to the initial state z_0 and disturbance $u(\cdot)$. The states $x(\tau_i)$, $i \in [1 : m-1]$ are measured with an error. The results of these measurements, the vectors $\xi_i^h \in \mathbb{R}^n$, satisfy the inequalities

$$\left| x(\tau_i) - \xi_i^h \right|_n \leqslant h. \tag{2}$$

Here, $h \in (0, 1)$ is the measurement accuracy; the symbol $|\cdot|_n$ stands for the Euclidean norm in the space $\mathbb{R}^n$. Our goal is to design an algorithm for approximate reconstruction of the unknown disturbance $u(\cdot)$ on the basis of inaccurate measurements of $x(\tau_i)$. In other words, the task is, given the current measurements of $x(\tau_i)$, to design a feedback algorithm that generates in real time a function $u^h = u^h(\cdot)$ that approximates the disturbance $u(\cdot)$ (in the metric of the space $L_2(T; \mathbb{R}^r)$). It will be seen from the description of the reconstruction algorithm that this function can be treated as a control of a suitable auxiliary system.

The problem described above belongs to the class of dynamical reconstruction problems. The analogous problems attract a great attention in recent years. There are a lot of monographs and reviews devoted to reconstruction (identification) problems, including problems for dynamical systems. List only some of them. The monograph by [12] contains the introduction to identification theory. The monograph by [15] focuses on the method of Poisson moment functionals and their application to identification theory. An algorithm for estimating a nonstationary input acted upon a linear system is considered in the monograph by [2]. The monograph by [1] concentrates on the questions of reconstructing unknown characteristics of distributed systems. The classical monograph by [5] is popular up to now; the emphasis in it is on different classes of recurrent methods for the identification of nonstationary objects, including their theoretical analysis and experimental verification.

One of approaches to solving dynamical reconstruction problems was developed in [4, 6, 8–11, 13, 14]. This approach is based on the methods of feedback control theory [3] and methods of ill-posed problems. In the case when the disturbance $u(\cdot)$ are subject to a priori constraints and all phase coordinates of the system (1) are measured, the problem in question can be solved on the base of constructions of [13, 14]. In the present paper we consider the case when only a part of coordinates are measured. In addition, we assume that instantaneous constraints on the disturbance are absent. Accordingly, $u(\cdot)$ is assumed to be a square integrable function. We design a solving algorithm. In connection with the fact that only the part $x(\tau_i)$ of the system's phase state $\{x(\tau_i), y(\tau_i)\}$ is measured, we need an additional block: the block of dynamical reconstruction of the unknown coordinate y. This block is considered as a provider of the information on the current phase state of system (1). The information is transferred to the block of disturbance reconstruction; the latter forms an approximation to $u(\cdot)$ using feedback laws.

Other dynamical reconstruction problems with solution algorithms based on a suitable modification of the extremal shift method were discussed, for example, in [4,6,8–11,14]. More specifically, the case when "all" phase state are measured was considered in [6,8,10]. The case when a part of the phase state are measured was discussed in [4,9,14].

2. Method for solving the problem

Let us describe a method for solving the problem under consideration. As was noted above, the method is based on construction of feedback control theory. The dynamical reconstruction problem is replaced by the problem of feedback control of a suitable dynamical system. In our case, the latter problem consists of two control blocs for systems of difference form.

For any $h \in (0,1)$, let us fixed a family of partitions of the interval T by control moments of time $\tau_{h,i}$:

$$\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h}, \quad \tau_{h,0} = 0, \quad \tau_{h,m_h} = \vartheta, \quad \tau_{h,i+1} = \tau_{h,i} + \delta(h), \quad \delta(h) \in (0,1). \tag{3}$$

The first (auxiliary) block involves a controlled system and a feedback control law U that generates a control $u^h(\cdot)$. The dynamics of the system is described by the vector differential equation

$$\dot{w}_1^h(t) = A\xi_i^h + By_0 + f_1(\tau_i) + u^h(t) \quad \text{for a.a.} \quad t \in [\tau_i, \tau_{i+1}) (i \in [0 : m_h]) \tag{4}$$

with the initial state $w^h(0) = x_0$. Here, the control $u^h(\cdot)$ is defined by the formula

$$u^h(t) = u_i^h = U(\tau_i, \xi_i^h, w_1^h(\tau_i)) \quad \text{for a.a. } t \in [\tau_i, \tau_{i+1}) \quad (i \in [0 : m_h - 1]), \tag{5}$$

where ξ_i^h is the result of measuring the coordinate $x(\tau_i)$ (see (2)). For $i = 0$, we set $\xi_0^h = x_0$. The law $U(\cdot,\cdot,\cdot) : T \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^N$ is constructed in such a way that the control $u^h(\cdot)$ approximates (in the metric of the space of continuous functions) the unobserved component $y(\cdot)$ under appropriate relations between parameters h and $\delta(h)$. In this case, system (4) and the control law (5) for system form the block of dynamical reconstruction of the component $y(\cdot)$.

The second (basic) block, the block of dynamical reconstruction of unknown disturbance. This block contains a the system

$$\dot{y}^h(t) = C\xi_i^h + Dy^h(t) + Ev^h(t) + f_2(t) \qquad \text{for a.a.} \quad t \in [\tau_i, \tau_{i+1}) \quad (i \in [0 : m_h - 1]) \tag{6}$$

with the initial state $y^h(0) = y_0$ and a control law $V(\cdot,\cdot,\cdot) : T \times \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}^r$ that generates the control $v^h(\cdot)$. The law V is constructed in such a way that, for suitable compatible parameters h and δ, the control $v^h(\cdot)$ of the form

$$v^h(t) = v_i^h = V(\tau_i, u_i^h, y^h(\tau_i)) \quad \text{for a.a.} \quad t \in [\tau_i, \tau_{i+1}) \quad (i \in [0 : m_h - 1]) \tag{7}$$

approximates the unknown disturbance.

It should be noted that one and the same solution of system (1) can be derived by multiple disturbances. Let $U(z(\cdot))$ is the set of all function $u(\cdot) \in L_2(T; \mathbb{R}^r)$ generating the solution $z(\cdot) = \{x(\cdot), y(\cdot)\}$ of equation (1). Let $u_*(\cdot)$ means an element of the set $U(z(\cdot))$ of minimal $L_2(T; \mathbb{R}^r)$-norm ; i.e.

$$u_*(\cdot) = \arg \min_{u(\cdot) \in U(z(\cdot))} |u(\cdot)|_{L_2(T;\mathbb{R}^r)}.$$

Note that the set $U(z(\cdot))$ is convex and closed in the space $L_2(T; \mathbb{R}^r)$. Therefore, the element $u_*(\cdot)$ is unique. In compliance with the conventional in the theory of ill-posed problems approach, we reconstruct $u_*(\cdot)$.

3. Solving algorithm

Let us describe the solution algorithm of the problem under consideration. Consider a family Δ_h (3) and two functions $\alpha(h) : (0, 1) \to (0, 1)$ and $\alpha_1(h)$: $(0, 1) \to (0.1)$.

Let $\mathcal{Y}(t)$ be the fundamental matrix of the system $\dot{y}(t) = Dy(t)$. Then, the inequality

$$|\mathcal{Y}(t)| \leqslant \exp\{\chi t\}, \quad t \geqslant 0 \tag{8}$$

is valid. Here, $\chi = |D|$, the symbol $|\cdot|$ stands for the Euclidean norm of a matrix.

Before starting the work of the algorithm, we fix the value $h \in (0, 1)$, numbers $\alpha_1 = \alpha_1(h)$ and $\alpha = \alpha(h)$ and a partition $\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h}$ of form (3). The work of the algorithm is decomposed into $m - 1$ ($m = m_h$) steps. At the i-th step carried out during the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$, the following actions take place. First, at the time τ_i, the vectors u_i^h and v_i^h are calculated by formulas (5) and (7), in which

$$\begin{aligned} U\left(\tau_i, \xi_i^h, w_1^h(\tau_i)\right) &= -\alpha_1^{-1}\left[w_1^h(\tau_i) - \xi_i^h\right], \\ V\left(\tau_i, u_i^h, y^h(\tau_i)\right) &= -\alpha^{-1}\exp\left\{-2\chi\tau_{i+1}\right\} E'\left(y^h(\tau_i) - y_0 - B^+ u_i^h\right). \end{aligned} \tag{9}$$

Here, the prime means transposition, the symbol B^+ stands for the pseudo inverse matrix for the matrix B. Then, for all $t \in \delta_i$, the control $u^h(t)$ of form (5), (9) is taken as the input of system (4), while a control $v^h(t)$ of form (7), (9) is taken as the input of system (6). As a result, under the action of such controls, system (4) passes from the state $w_1^h(\tau_i)$ to the state $w_1^h(\tau_{i+1})$, while system (6) passes from the state $y^h(\tau_i)$ to the state $y^h(\tau_{i+1})$. The procedure stops at time ϑ.

Let us show that feedbacks $V(\cdot,\cdot,\cdot)$ and $U(\cdot,\cdot,\cdot)$ of form (9) solve the problem.

Before proceeding to the proof of the theorem, we present two lemmas used below.

Lemma 1 [6, p. 47] *Let $u(\cdot) \in L_\infty(T_*;\mathbb{R}^n)$ and $v(\cdot) \in W(T_*;\mathbb{R}^n)$, $T_* = [a,b]$, $-\infty < a < b < +\infty$,*

$$\left|\int_a^t u(\tau)\mathrm{d}\tau\right|_n \leqslant \varepsilon, \quad |v(t)|_n \leqslant K \quad \forall\, t \in T_*.$$

Then, for all $t \in T_*$, the inequality

$$\left|\int_a^t (u(\tau), v(\tau))\mathrm{d}\tau\right|_+ \leqslant \varepsilon(K + \mathrm{var}(T_*; v(\cdot)))$$

is valid.

Here, the symbol $\mathrm{var}(T_*; v(\cdot))$ means the variation of the function $v(\cdot)$ over the interval T_*, the symbol $(\cdot,\cdot)$ means the scalar product in the corresponding finite-dimensional Euclidean space, the symbol $|\cdot|_+$ means the absolute value of a number, and the symbol $W(T_*;\mathbb{R}^n)$ means the set of functions $y(\cdot) : T_* \to \mathbb{R}^n$ of bounded variation.

Lemma 2 [7] *Let a nonnegative function $\phi(t)$, $t \in T$, satisfy the inequalities*

$$\phi(\tau_{i+1}) \leqslant \phi(\tau_i)(1 + p\delta) + \int_{\tau_i}^{\tau_{i+1}} |G(\tau)|_+ \mathrm{d}\tau$$

for all $i \in [0 : m-1]$, where $\tau_i \in \Delta$, $\delta = \tau_{i+1} - \tau_i$, $p = \mathrm{const} > 0$, $G(\cdot) \in L_\infty(T;\mathbb{R})$. Then, the inequalities

$$\phi(\tau_i) \leqslant \left(\phi(0) + \int_0^{\tau_i} |G(\tau)|_+ \mathrm{d}\tau\right)\exp(p\tau_i), \quad i \in [0 : m],$$

take place.

Theorem 1 *Let $\delta(h) = h$, $\alpha_1(h) = h^{1/2}$, $\alpha(h) \to 0$ and $h\alpha^{-4}(h) \to 0$ as $h \to 0$. Let also $N \leqslant n$ and $\operatorname{rank} B = N$. Then there exist functions $\nu_1(\cdot) : (0,1) \mapsto [0,+\infty)$, $\nu_2(\cdot) : (0,1) \mapsto [0,+\infty)$, $r_1(\cdot) : (0,1) \mapsto [0,+\infty)$ and $r_2(\cdot) : (0,1) \mapsto [0,+\infty)$ such that $\nu_1(h) \to 0$, $\nu_2(h) \to 0$, $r_1(h) \to 1$, $r_2(h) \to 0$ as $h \to 0$ and the inequalities*

$$\sup_{t\in T} \left|B^+ u^h(t) + y_0 - y(t)\right|_N \leqslant \nu_1(h), \tag{10}$$

$$\max_{i\in[0:m_h]} \left|y(\tau_{h,i}) - y^h(\tau_{h,i})\right|_N \leqslant \nu_2(h), \tag{11}$$

$$\int_0^{\vartheta} \left|B^+ v^h(\tau)\right|_r^2 \mathrm{d}\tau \leqslant r_1(h) \int_0^{\vartheta} |u_*(\tau)|_r^2 \mathrm{d}\tau + r_2(h) \tag{12}$$

are fulfilled for any disturbance $u(\cdot)$, any $h \in (0,1)$, any family Δ_h (see (3)), any realizations $v^h(\cdot)$ and $u^h(\cdot)$ of feedbacks $V(\cdot,\cdot,\cdot)$ and $U(\cdot,\cdot,\cdot)$ of form (9), any trajectory of real system (1) $z(\cdot) = z(\cdot; z_0, u(\cdot))$ (i.e., solution of (1)), any trajectory $w_1^h(\cdot)$ and $y^h(\cdot)$ of systems (4) and (6) corresponding to the feedbacks V and U, and any measurement ξ_i^h with property (2).

Proof. The proof of the theorem consists of *three steps*. At the first step, we prove inequality (10). At the second one, we estimate the change of the function $\lambda_h(\tau_i)$, where $\lambda_h(t) = \exp(-2\chi t)|y^h(t) - y(t)|_N^2$. At the third step, we prove the inequalities (11) and (12).

Step 1. Introduce the new function

$$Y(t) = y(t) - y_0\,.$$

Then first subsystem of system (1) takes the form

$$\dot{x}(t) = Ax(t) + BY(t) + By_0 + f_1(t).$$

In virtue of condition $\dot{f}_1(\cdot) \in L_\infty(T; \mathbb{R}^n)$ and the inequalities (2), we conclude that the inequalities

$$\left|Ax(t) + By_0 + f_1(t) - A\xi_i^h - By_0 - f_1(\tau_i)\right|_n \leqslant M_1(h+\delta)$$

are fulfilled for a.a. $t \in \delta_i$ and all $i \in [0 : m-1]$. Note that $Y(\cdot) \in W^{1,\infty}(T; \mathbb{R}^N)$ and $Y(0) = 0$. Then, from Theorem 2 [8] we derive the estimate

$$\left|u^h(t) + B(y_0 - y(t))\right|_n = \left|u^h(t) - BY(t)\right|_n \leqslant M_2 h_1\,. \tag{13}$$

Here,

$$h_1 = \alpha_1 + (h+\delta)\alpha_1^{-1}, \quad \alpha_1 = \alpha_1(h), \quad \delta = \delta(h),$$

$M_1 > 0$ and $M_2 > 0$ are some constants. Inequality (10) follows from inequality (13).

Step 2. We estimate the change of the function

$$\varepsilon_h(t) = \lambda_h(t) + \alpha \int_0^t \left\{ |v^h(\tau)|_r^2 - |u_*(\tau)|_r^2 \right\} \mathrm{d}\tau. \tag{14}$$

In virtue of the Cauchy formula, we conclude that the equalities

$$\begin{aligned} y(t) &= \mathcal{Y}(t-\tau_i)y(\tau_i) + \int_{\tau_i}^t \mathcal{Y}(t-\tau)\left\{Cx(\tau) + Eu_*(\tau) + f_2(\tau)\right\} \mathrm{d}\tau, \\ y^h(t) &= \mathcal{Y}(t-\tau_i)y^h(\tau_i) + \int_{\tau_i}^t \mathcal{Y}(t-\tau)\left\{C\xi_i^h + Ev^h(\tau) + f_2(\tau)\right\} \mathrm{d}\tau \end{aligned} \tag{15}$$

are fulfilled for all $t \in \delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$. By using equalities (15), it is easily seen that, for all $i \in [0 : m-1]$, the estimate

$$\begin{aligned} \varepsilon_h(\tau_{i+1}) \leqslant{} & \exp\{-2\chi\tau_{i+1}\}\left|y^h(\tau_i) - y(\tau_i)\right|_n^2 + \lambda_i + \mu_i \\ & + \alpha \int_0^{\tau_{i+1}} \left\{ |v^h(\tau)|_r^2 - |u_*(\tau)|_r^2 \right\} \mathrm{d}\tau \end{aligned} \tag{16}$$

is valid. Here,

$$\lambda_i = 2\left(S_i, \int_{\tau_i}^{\tau_{i+1}} \mathcal{Y}(\tau_{i+1}-\tau)\left[E\{v^h(\tau) - u_*(\tau)\} + C\left\{\xi_i^h - x(\tau)\right\}\right] \mathrm{d}\tau \right), \tag{17}$$

$$\begin{aligned} \mu_i = \delta \exp\{-2\chi\tau_{i+1}\} \int_{\tau_i}^{\tau_{i+1}} \Big| \mathcal{Y}(\tau_{i+1}-\tau)\Big\{ & E\left(v^h(\tau) - u_*(\tau)\right) \\ & + C\left(\xi_i^h - x(\tau)\right)\Big\}\Big|_N^2 \mathrm{d}\tau, \end{aligned} \tag{18}$$

$$S_i = \exp\{-2\chi\tau_{i+1}\}\mathcal{Y}(\tau_{i+1}-\tau_i)\left\{y^h(\tau_i) - y(\tau_i)\right\}, \quad \delta = \tau_{i+1} - \tau_i\,. \tag{19}$$

Using the inequality $\exp\{-2\chi\delta\} \leqslant 1$ and (16), we derive the estimate

$$\varepsilon_h(\tau_{i+1}) \leqslant \varepsilon_h(\tau_i) + \lambda_i + \mu_i + \alpha \int_{\tau_i}^{\tau_{i+1}} \left\{ |v^h(\tau)|_r^2 - |u_*(\tau)|_r^2 \right\} \mathrm{d}\tau. \tag{20}$$

Consider the function λ_i (see (17)). Taking into account (8), (2), and (19), one can show that the inequalities

$$\begin{aligned} &|(S_i, \mathcal{Y}(\tau_{i+1}-\tau)Eu) - \exp\{-2\chi\tau_{i+1}\}(s_i, Eu)| \\ &\qquad \leqslant C_0\delta\lambda_h^{1/2}(\tau_i)|Eu|_N + C_1 h_1 |Eu|_N \quad \forall u \in \mathbb{R}^r \end{aligned} \tag{21}$$

hold. Here, $s_i = y^h(\tau_i) - y_0 - B^+ u_i^h$. Note that $f_1(\cdot) \in L_\infty(T;\mathbb{R}^n)$. Therefore,

$$\left|\xi_i^h - x(t)\right|_n \leqslant h + \int\limits_{\tau_i}^{\tau_{i+1}} |\dot{x}(t)|_n \mathrm{d}t \leqslant C_2(h+\delta), \qquad t \in \delta_i. \tag{22}$$

In turn, in virtue of (17), (22) and (21), we obtain the estimate

$$\lambda_i \leqslant 2\exp\{-2\chi\tau_{i+1}\}\int\limits_{\tau_i}^{\tau_{i+1}} \left(s_i, E\{v^h(\tau) - u_*(\tau)\}\right)\mathrm{d}\tau + \rho_i + \rho_i^{(1)}, \tag{23}$$

where

$$\begin{aligned} \rho_i^{(1)} &= 2\left|S_i, \int\limits_{\tau_i}^{\tau_{i+1}} \mathcal{Y}(\tau_{i+1}-\tau_i)C\left\{\xi_i^h - x(\tau)\right\}\mathrm{d}\tau\right| \\ &\leqslant C_3(h+\delta)\delta\lambda_h^{1/2}(\tau_i) \leqslant C_4\delta^{3/2}\lambda_h(\tau_i) + C_4\delta^{1/2}(h+\delta)^2, \end{aligned} \tag{24}$$

It is easily seen that the inequalities

$$\delta\lambda_h^{1/2}(\tau_i)\int\limits_{\tau_i}^{\tau_{i+1}}\left|E\{v^h(\tau)-u_*(\tau)\}\right|_N \mathrm{d}\tau \leqslant \delta^2\lambda_h(\tau_i) + |E|^2\delta\int\limits_{\tau_i}^{\tau_{i+1}}\left\{|v^h(\tau)|_r^2 + |u_*(\tau)|_r^2\right\}\mathrm{d}\tau,$$

$$h_1\int\limits_{\tau_i}^{\tau_{i+1}}\left|E\{v^h(\tau)-u_*(\tau)\}\right|_N \mathrm{d}\tau \leqslant h_1^2\delta^{1/2-\varepsilon} + |E|^2\delta^{1/2+\varepsilon}\int\limits_{\tau_i}^{\tau_{i+1}}\left\{|v^h(\tau)|_r^2 + |u_*(\tau)|_r^2\right\}\mathrm{d}\tau$$

holds for any $\varepsilon \in (0, 1/2)$.

$$\rho_i = C_5\left\{\delta\lambda_h^{1/2}(\tau_i) + h_1\right\}\int\limits_{\tau_i}^{\tau_{i+1}}\left|E\{v^h(\tau)-u_*(\tau)\}\right|_N \mathrm{d}\tau.$$

Therefore,

$$\rho_i \leqslant C_6 \left\{ \delta^2 \lambda_h(\tau_i) + h_1^2 \delta^{1/2-\varepsilon} + \delta^{1/2+\varepsilon} \int\limits_{\tau_i}^{\tau_{i+1}} \left\{ \left| v^h(\tau) \right|_r^2 + |u_*(\tau)|_r^2 \right\} \mathrm{d}\tau \right\}. \tag{25}$$

Then, we get (see (18), (22))

$$\mu_i \leqslant C_7 \delta \exp\{-2\chi\tau_{i+1}\} \int\limits_{\tau_i}^{\tau_{i+1}} \left\{ |v^h(\tau)|_r^2 + |u_*(\tau)|_r^2 \right\} \mathrm{d}\tau + C_7 \delta^2 (h+\delta)^2$$

$$\leqslant C_8 \delta^{1/2+\varepsilon} \int\limits_{\tau_i}^{\tau_{i+1}} \left\{ |v^h(\tau)|_r^2 + |u_*(\tau)|_r^2 \right\} \mathrm{d}\tau + C_7 \delta^2 (h+\delta)^2. \tag{26}$$

Note that the vector v_i^h (see (7), (9)) is found from the condition

$$v_i^h = \arg\min \left\{ 2 \exp\{-2\chi\tau_{i+1}\}(s_i, Ev) + \alpha |v|_r^2 : v \in \mathbb{R}^r \right\}. \tag{27}$$

Consequently, from (23) and (27) we derive the inequality

$$\lambda_i + \alpha \int\limits_{\tau_i}^{\tau_{i+1}} \left\{ |v^h(\tau)|_r^2 - |u_*(\tau)|_r^2 \right\} \mathrm{d}\tau \leqslant \rho_i + \rho_i^{(1)}. \tag{28}$$

Hence, in virtue of (20), (24)-(26) and (28), we obtain for $\varepsilon \in (0, 1/2)$

$$\varepsilon_h(\tau_{i+1}) \leqslant \varepsilon_h(\tau_i) + C_9 \left\{ \delta^{3/2} \lambda_h(\tau_i) + h_1^2 \delta^{1/2-\varepsilon} \right.$$

$$\left. + \delta^{1/2+\varepsilon} \int\limits_{\tau_i}^{\tau_{i+1}} \left\{ |v^h(\tau)|_r^2 + |u_*(\tau)|_r^2 \right\} \mathrm{d}\tau + \delta^{1/2}(h+\delta)^2 \right\}, \tag{29}$$

i.e., (see (14))

$$\lambda_h(\tau_{i+1}) \leqslant (1 + C_{10}\delta)\lambda_h(\tau_i) + C_{11} \left\{ h_1^2 \delta^{1/2-\varepsilon} \right.$$

$$\left. + \delta^{1/2+\varepsilon} \int\limits_{\tau_i}^{\tau_{i+1}} \left\{ |v^h(\tau)|_r^2 + |u_*(\tau)|_r^2 \right\} \mathrm{d}\tau + \delta^{1/2}(h+\delta)^2 \right\} + \alpha \int\limits_{\tau_i}^{\tau_{i+1}} |u_*(\tau)|_r^2 \mathrm{d}\tau.$$

The rule for finding of the control v_i^h implies the inequalities

$$|v_i^h|_N^2 \leqslant 2|E|^2 \exp\{-2\chi(\tau_{i+1}+\delta)\}\left(\lambda_h(\tau_i)+h_1^2\right)\alpha^{-2}$$
$$\leqslant C_{12}\left(\lambda_h(\tau_i)+h_1^2\right)\alpha^{-2}. \tag{30}$$

In addition, we have

$$\lambda_h(0)=0. \tag{31}$$

Using inequalities (30) and

$$\delta^{1/2+\varepsilon}\int\limits_{\tau_i}^{\tau_{i+1}}|v^h(\tau)|_r^2\,\mathrm{d}\tau \leqslant C_{12}\delta^{3/2+\varepsilon}\left(\lambda_h(\tau_i)+h_1^2\right)\alpha^{-2}$$
$$= C_{12}\delta^{3/2+\varepsilon}\alpha^{-2}\lambda_h(\tau_i)+C_{12}\delta^{3/2+\varepsilon}\alpha^{-2}h_1^2,$$

we obtain (for $\delta(h)\in(0,1)$, $\delta^{1/2+\varepsilon}(h)\alpha^{-2}(h)\leqslant const.$) the relation

$$\lambda_h(\tau_{i+1})\leqslant(1+C_{13}\delta)\lambda_h(\tau_i)$$
$$+\ C_{14}\left\{h_1^2\delta^{1/2-\varepsilon}+\left(\alpha+\delta^{1/2+\varepsilon}\right)\int\limits_{\tau_i}^{\tau_{i+1}}|u_*(\tau)|_r^2\,\mathrm{d}\tau+\delta^{1/2}(h+\delta)^2\right\}.$$

Hence, taking into account (31) and Lemma 2, we get

$$\lambda_h(\tau_{i+1})\leqslant C_{15}\left(\alpha+\delta^{1/2+\varepsilon}+h_1\delta^{-1/2-\varepsilon}+\delta^{-1/2}(h+\delta)^{1/2}\right). \tag{32}$$

Step 3. In virtue of (30) and (32), we have the inequalities

$$I_i=\delta^{1/2+\varepsilon}\int\limits_0^{\tau_{i+1}}|v^h(\tau)|_r^2\,\mathrm{d}\tau\leqslant C_{12}\delta^{3/2+\varepsilon}\alpha^{-2}\sum_{j=0}^{i}\left(\lambda_h(\tau_j)+h_1^2\right)$$
$$\leqslant C_{16}\delta^{1/2+\varepsilon}\alpha^{-2}\left(\alpha+\delta^{1/2+\varepsilon}+h_1^2\delta^{1/2-\varepsilon}+\delta^{-1/2}(h+\delta)^2\right). \tag{33}$$

It is easily seen that the inequality

$$h_1^2\delta^{-1/2-\varepsilon}\leqslant 2\alpha_1^2\delta^{-1/2-\varepsilon}+4\left(h^2+\delta^2\right)\alpha_1^{-2}\delta^{-1/2-\varepsilon} \tag{34}$$

is valid. Assume $\alpha_1=\delta^{1/4+\varepsilon}$. Then $\alpha_1^2\delta^{1/2+\varepsilon}=\delta^{1+3\varepsilon}$. Hence, in virtue of this equality, we obtain

$$\alpha_1^2\delta^{-1/2-\varepsilon}=\delta^{\varepsilon},\qquad \delta^2\alpha_1^{-2}\delta^{-1/2-\varepsilon}=\alpha_1^{-2}\delta^{3/2-\varepsilon}=\delta^{1-3\varepsilon}. \tag{35}$$

Moreover, $\delta^{1-3\varepsilon} \to 0$ if $\delta \to 0$ and $\varepsilon \in (0, 1/3)$. So, if we assume

$$\varepsilon = 1/4, \qquad \delta = h, \qquad \alpha_1 = \delta^{1/4+\varepsilon} = h^{1/2},$$

then, in virtue of (34) and (35), we derive the estimate

$$\begin{aligned} h_1^2\delta^{-1/2-\varepsilon} &\leqslant C_{17}\left(h^2\delta^{-1-3\varepsilon} + \delta^{\varepsilon} + \delta^{1-3\varepsilon}\right) \\ &\leqslant C_{18}\left(h^2\delta^{-7/4} + \delta^{1/4}\right) \leqslant C_{19}h^{1/4}. \end{aligned} \tag{36}$$

In turn, by the use of (33) and (36), we deduce that

$$I_i \leqslant C_{20}h^{3/4}\alpha^{-2}\left(\alpha + h^{3/4} + h^{1/4}\right) \leqslant C_{21}h^{3/4}\alpha^{-2}\left(\alpha + h^{1/4}\right). \tag{37}$$

Note that

$$h^{1/4} \leqslant C_{22}\alpha(h) \qquad \text{for} \quad h \in (0, 1). \tag{38}$$

Then, from (36), (38) and (32) we have

$$\lambda_h(\tau_i) \leqslant C_{22}(\alpha + h^{1/4}) \leqslant C_{23}\alpha, \qquad i \in [0 : m]. \tag{39}$$

Thus inequality (11) follows from (39) ($\nu_2(h) = const\alpha^{1/2}(h)$). Next, summing (29) over i from 0 to j, in virtue of (31), (37) and (39), we have

$$\begin{aligned} \varepsilon_h(\tau_{i+1}) \leqslant C_{24}\Bigg\{ & h^{1/4} + h^{1/2}\left(\alpha + h^{1/4}\right) \\ & + h^{3/4}\int_0^{\tau_{i+1}} |u_*(\tau)|_r^2 \,\mathrm{d}\tau + h\alpha^{-2} + h^{3/4}\alpha^{-1}\Bigg\}. \end{aligned} \tag{40}$$

In this case, from (40) we get

$$\int_0^{\tau_{i+1}} |v^h(\tau)|_r^2 \,\mathrm{d}\tau \leqslant k_1(h)\int_0^{\tau_{i+1}} |u_*(\tau)|_r^2 \,\mathrm{d}\tau + k_2(h), \quad i \in [0 : m{-}1], \tag{41}$$

where

$$\begin{aligned} k_1(h) &= 1 + d_1h^{3/4}\alpha^{-1}(h), \\ k_2(h) &= d_2\left(h^{1/4}\alpha^{-1}(h) + h\alpha^{-3}(h) + h^{3/4}\alpha^{-2}(h) + h^{1/2}\right). \end{aligned}$$

It is easily seen that the inequalities

$$k_1(h) \leqslant r_1(h) = 1 + d_3\alpha^2(h), \qquad k_2(h) \leqslant r_2(h) = d_4h^{1/4}\alpha^{-1}(h) \tag{42}$$

holds. The inequality (12) follows from (41) and (42). The theorem is proved. □

Theorem 2 *Let the conditions of Theorem 1 be fulfilled. Then the convergence*

$$v^h(\cdot) \to u_*(\cdot) \quad in \quad L_2(T;\mathbb{R}^r) \quad as \quad h \to 0$$

takes place.

The assertion of Theorem 2 follows Theorem 1 and Theorem 1.2.1 [6, p. 23].

Under some additional conditions, we can obtain the algorithm's convergence rate (see Theorem 3 below). Note that, under the condition of Theorem 1, the solution's component $y(\cdot)$ is uniquely determined by the component $x(\cdot)$ from the first subsystem of (1).

Theorem 3 *Let the conditions of Theorem 1 be fulfilled. Let also* $r \leqslant N$, rank $E = r$, *and* $u(\cdot) \in W(T;\mathbb{R}^r)$. *Then the following inequality is valid:*

$$\left| u_*(\cdot) - v^h(\cdot) \right|^2_{L_2(T;\mathbb{R}^r)} \leqslant K_0 \left\{ \alpha^{1/2}(h) + h^{1/4}\alpha^{-1}(h) \right\}, \tag{43}$$

Here K_0 is a constant not depending on h and α.

Proof. By the use of (15), it is easily seen that the inequalities

$$\lambda_h^{1/2}(t) \leqslant c_1 \left\{ \lambda_h^{1/2}(\tau_i) + \int\limits_{\tau_i}^{t} \left(|v_i^h|_N + |y(\tau)|_N \right) \mathrm{d}\tau \right\} \tag{44}$$

are fulfilled for $t \in [\tau_i, \tau_{i+1}]$, $i \in [0 : m-1]$. Note that $h_1 = h^{1/2}$. In turn, from (38) and (39) we obtain

$$\int\limits_{\tau_i}^{\tau_{i+1}} |v_i^h|_N \,\mathrm{d}\tau \leqslant c_2 \delta \alpha^{-1} \left(h_1 + \lambda_h^{1/2}(\tau_i) \right) \leqslant c_3 h \alpha^{-1/2}. \tag{45}$$

In this case, from (44), (38) and (45), in virtue of (39) we get for $t \in [\tau_i, \tau_{i+1}]$

$$|z_h(t)|_n \leqslant c_4 \left(\alpha^{1/2}(h) + h\alpha^{-1/2}(h) \right), \tag{46}$$

where $z_h(t) = w^h(t) - x(t)$. Let the symbol $\widetilde{E}$ stand for an $r \times r$ matrix that consists of r columns of matrix E and has the rank r. In turn, in virtue of (1), (6), (46), and (22), the estimate

$$\left| \int\limits_{t_1}^{t_2} \widetilde{E} \left(u_*(t) - v^h(t) \right) \mathrm{d}t \right|_r \leqslant c_5 \left\{ |z_h(t_2) - z_h(t_1)|_n + \int\limits_{t_1}^{t_2} |z_h(t)|_n \mathrm{d}t \right\} + c_6 h$$
$$\leqslant c_7 \alpha^{1/2}(h)$$

is fulfilled for all $t_1, t_2 \in T$, $t_1 < t_2$. By the use of inequality (41), from Lemma 1 we get the relations

$$\left|u_*(\cdot) - v^h(\cdot)\right|^2_{L_2(T;\mathbb{R}^r)} \leqslant 2|u_*(\cdot)|^2_{L_2(T;\mathbb{R}^r)} - 2\int_0^{\vartheta}(u_*(\tau), v^h(\tau))\mathrm{d}\tau + r_2(h)$$

$$+ (r_1(h) - 1)\int_0^{\vartheta}|u_*(\tau)|^2_r \mathrm{d}\tau = 2\int_0^{\vartheta}\left(\left(\widetilde{E}^{-1}\right)' u_*(\tau), \widetilde{E}\left(u_*(\tau) - v^h(\tau)\right)\right)\mathrm{d}\tau$$

$$+ r_2(h) + (r_1(h) - 1)\int_0^{\vartheta}|u_*(\tau)|^2_r \mathrm{d}\tau \leqslant c_8\left\{\alpha^{1/2} + r_2(h) + (r_1(h) - 1)\right\}. \quad (47)$$

From (47) and (12) we derive the inequality (43). The theorem is proved. □

4. Example

A material point of unit mass moves along a line under the action of a tractive force. The gravity force is ignored. The travel of the point is inaccurately measured at discrete, frequent enough, times. It is required to design an algorithm of reconstructing (in real time mode) the unknown force. According to the second Newton law, the motion is described by the equation

$$\ddot{x}(t) = u(t), \qquad t \in [0, \vartheta], \quad (48)$$

where $u(t)$ is the outer force, $x(t)$ is the travel of the point. Let the initial state x_0 and initial velocity y_0 be known. Assuming $\dot{x}(t) = y(t)$, rewrite equation (48) in the form of system (1)

$$\begin{aligned} \dot{x}(t) &= y(t), \qquad & x(0) &= 0, \\ \dot{y}(t) &= u(t), \qquad & y(0) &= y_0 . \end{aligned} \quad (49)$$

In this case, $f_1(t) = f_2(t) = 0$ for $t \in [0, \vartheta]$. The equations of the auxiliary systems have the form of system (4)

$$\dot{w}_1^h(t) = u^h(t) + y_0, \qquad w_1^h(0) = x_0, \quad (50)$$

and system (6) is

$$\dot{y}^h(t) = v^h(t), \qquad y^h(0) = y_0 . \quad (51)$$

Systems (49)–(51) were solved using the Euler method with some integration step δ. At the moments $\tau_i = i\delta$, the values v_i^h and u_i^h are calculated by the formulas (see (5), (7) and (9))

$$u_i^h = \alpha_1^{-1}\left(\xi_i^h - w_1^h(\tau_i)\right),$$
$$v_i^h = \alpha^{-1}\exp\{-\tau_{i+1}\}\left(u_i^h + y_0 - y^h(\tau_i)\right).$$

In the numerical experiment, we set $\vartheta = 2$, $x_0 = 1$, $y_0 = 1$, $\delta = 0.003$, $\alpha_1 = 0.006$, $\alpha = 0.002$, $\xi_i^h = x(\tau_i) + h\cos(10t)$. The simulation results are presented in Figs. 1–4. Figures 1 and 2 correspond to the case $h = 0.01$, whereas Figs. 3

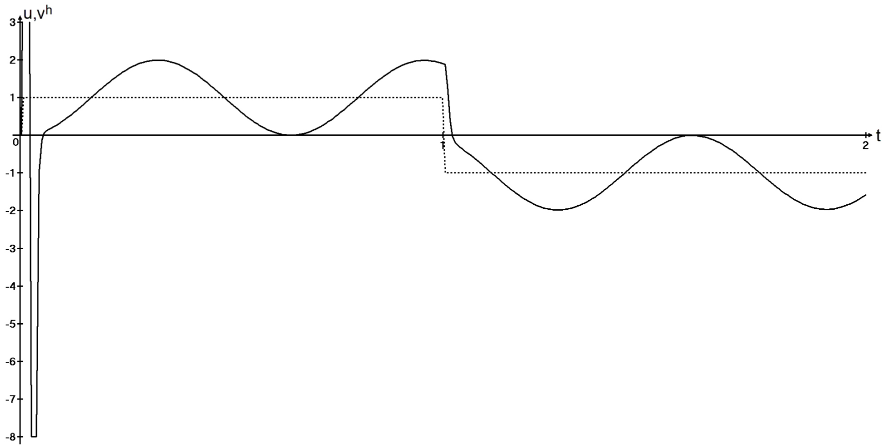

Figure 1: $h = 0.01$

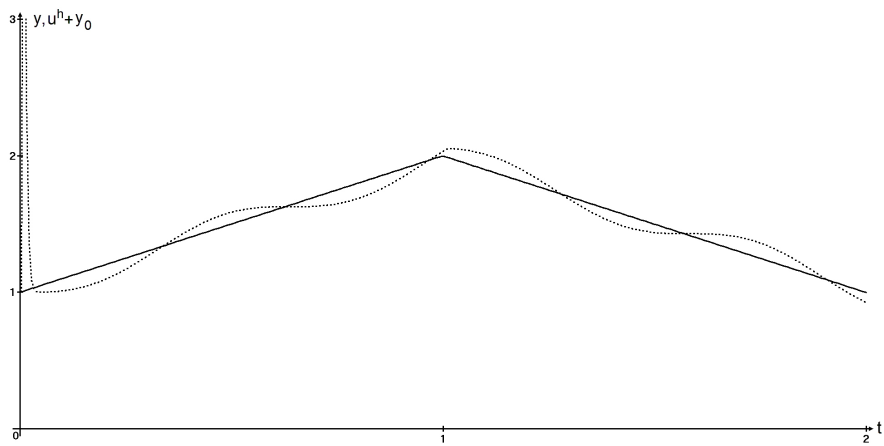

Figure 2: $h = 0.01$

and 4, to the case $h = 0.001$. In Figs. 1 and 3, the dashed lines represent the force $u(t) = 1$ if $t \in [0, 1]$, $u(t) = -1$ if $t \in (1, 2]$, while the solid lines represent the result of the work of the algorithm $v^h(t)$. In Figs. 2 and 4, the dashed lines represent $u^h(t) + y_0$, while the solid lines, $y(t)$. As we see from the Figs. 3 and 4, the solid and dashed lines virtually coincide.

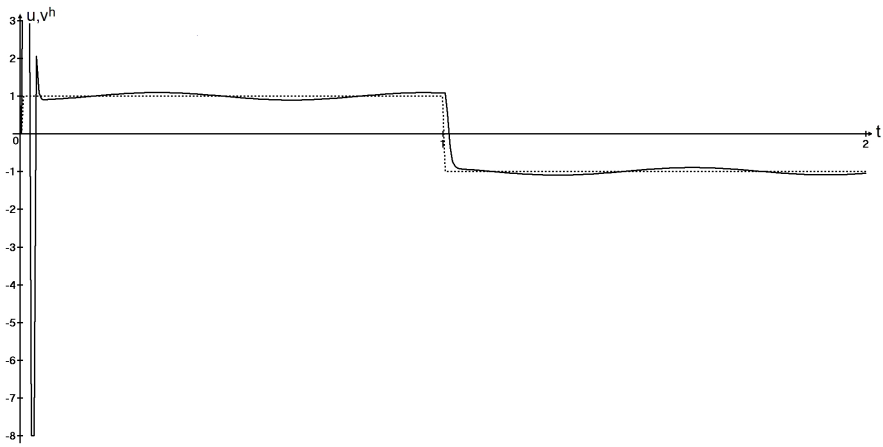

Figure 3: $h = 0.001$

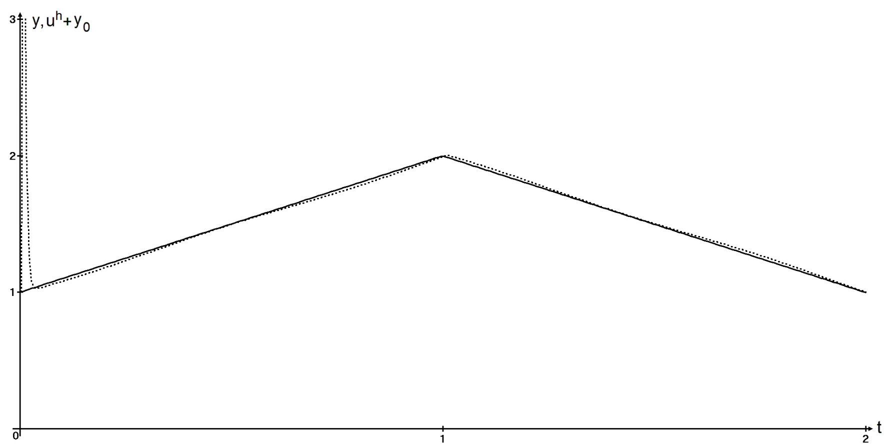

Figure 4: $h = 0.001$

5. Conclusion

The disturbance reconstruction problem for a system of linear differential equations is considered. The problem consists in designing an algorithm of dynamical reconstruction of an disturbance through measuring a part of system's phase coordinates. In the paper, the problem with two peculiarities is investigated. First, it is assumed that not all but a part of phase coordinates of the dynamical system is inaccurately measured at discrete, frequent enough, times. Second, as to the unknown disturbance, only the fact that this disturbance is an element of the space of square integrable functions (i.e., it may be unbounded) is known. Taking this feature of the problem into account, a solving algorithm that is stable to informational perturbations and computational errors is designed.

References

[1] H.T. Banks and K. Kunisch: *Estimation Techniques for Distributed Parameter Systems*, Boston: Birkhäuser, 1989.

[2] Y. Bar-Shalom and X.R. Li: *Estimation and Tracking: Principles, Techniques, and Software*, Boston: Artech House, 1993.

[3] N.N. Krasovskii and A.I. Subbotin: *Game-Theoretical Control Problems*, New York—Berlin: Springer Verlag, 1988.

[4] A. Kuklin, V. Maksimov, and N. Nikulina: On reconstructing unknown characteristics of a nonlineas system of differential equations, *Archives of Control Sciences*, **23**(2) (2015), 163–176.

[5] L. Ljung and T. Söderström: *Theory and Practice of Recursive Identification*, Massachusetts: M.I.T. Press, 1983.

[6] V.I. Maksimov: *Dynamical Inverse Problems for Distributed Systems*, Utrecht—Boston: VSP, 2002.

[7] V.I. Maksimov: The tracking of the trajectory of a dynamical system, *J. Appl. Math. Mech.*, **75** (2011), 667–674.

[8] V.I. Maksimov: An algorithm for reconstructing controls in a uniform metric, *J. Appl. Math. Mech.*, **77**(2) (2013), 212–219.

[9] V.I. Maksimov: Game control problem for a phase field equation, *J. Optim. Theory and Appl* (2016), JOTA-D-14-00400R1, DOI: 10.1007/s10957-015-0721-0.

[10] V.I. Maksimov: Dynamic reconstruction of system disturbances using inaccurate discrete measurements of phase coordinates, *J. Computer. Syst. Sc. Int.*, **57**(3) (2018), 358–373.

[11] V.I. Maksimov and F. Tröltzsch: Input reconstruction by feedback control for the Schlogl and Fitzhugh-Nagumo equations, *Int. J. Appl. Math. Comput. Sci.*, **30**(1) (2020), 5–22.

[12] J.P. Norton: *An Introduction to Identification*, London: Academic Press, 1986.

[13] Yu.S. Osipov and A.V. Kryazhimskii: *Inverse Problems for Ordinary Differential Equations: Dynamical Solutions*, London: Gordon and Breach, 1995.

[14] Yu.S. Osipov, A.V. Kryazhimskii, and V.I. Maksimov: *Methods of Dynamical Reconstruction of Inputs of Control systems*, UrO RAN, Ekaterinburg (in Russian), 2011.

[15] H. Unbehauen and G.P. Rao: *Identification of Continuous Systems*, Amsterdam: Elsevier, 1987.