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On the alternative stability criteria for positive systems

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Abstract. The paper discusses the stability problem for continuous time and discrete time positive systems. An alternative formulation of stability criteria for positive systems has been proposed. The results are based on a theorem of alternatives for linear matrix inequality (LMI) feasibility problem, which is a particular case of the duality theory for semidefinite programming problems.

1. Introduction

The main characteristic feature of positive systems is that for nonnegative initial conditions their state variables and outputs assume nonnegative values, provided the inputs are nonnegative [1–4]. Positive systems have been frequently encountered in practice since many quantities, such as for instance pressure, sugar concentration in blood, etc., take only nonnegative values. For these reasons, positive systems are frequently encountered in engineering [5–8], medicine and biology [9–15], economics etc. The stability and stabilizability are the key features we require from the dynamic systems, and the positive systems are not an exception in this regard. The stability problem for positive system has been considered in many papers, for example, [16–23]. Application of linear matrix inequalities to the stability checking has a long history, indeed, it can be traced back to the celebrated Lyapunov result on the stability of linear systems. More information on that can be found in [24]. The LMI framework has been successfully applied for checking stability of positive systems [24,25]. In [26] duality aspects of semidefinite programming are presented and the role they play in control theory. The duality results derived from optimization theory presented in [26] provide us with better insight into some problems of control theory. It turns out that some of ideas from [26] may be extended to study the positive systems. In this paper the problems of positive systems stability are addressed by means of LMIs (linear matrix inequalities), in particular, alternative formulations of stability criteria are proposed.

2. Preliminaries

2.1. Positive Systems. Let us denote by $\boldsymbol{R}^{m\times n}$ the set of all real matrices with m rows and n columns. Also let $\boldsymbol{R}^m := \boldsymbol{R}^{m\times 1}$.

Definition 1. [4] The matrix $A=[a_{ij}]\in\boldsymbol{R}^{n\times n}$ is called a Metzler matrix if its all off-diagonal entries are nonnegative, i.e., $a_{ij}\geq 0$ for $i\neq j$, $i,j=1,2,\ldots,n$.

The set of all $n\times n$ Metzler matrices will be denoted by $\boldsymbol{M}^n$.

Definition 2. [4] A matrix $A\in\boldsymbol{R}^{n\times m}$ is called nonnegative if its entries a_{ij} are nonnegative, i.e., $a_{ij}\geq 0$ for $i=1,\ldots,n$, $j=1,\ldots,m$.
The set of nonnegative $n\times m$ matrices is denoted $\boldsymbol{R}_+^{n\times m}$. Let us note, that nonnegative matrix $A\in\boldsymbol{R}^{n\times m}$ may have all entries equal to zero.

Let us consider the following linear time-continuous system

$$\dot{x}(t)=Ax(t)+Bu(t),\quad x(0)=x_0, \tag{1a}$$

$$y(t)=Cx(t), \tag{1b}$$

where $x(t)\in\boldsymbol{R}^n$, $y(t)\in\boldsymbol{R}^p$, and $u(t)\in\boldsymbol{R}^m$ are the state, output, and input vectors, respectively, and $A\in\boldsymbol{R}^{n\times n}$, $B\in\boldsymbol{R}^{n\times m}$, $C\in\boldsymbol{R}^{p\times n}$.

Definition 3. [4] The system (1) is called (internally) positive if for any $x_0\in\boldsymbol{R}_+^n$ and every input $u\in\boldsymbol{R}_+^m$ one has $x\in\boldsymbol{R}_+^n$ and $y\in\boldsymbol{R}_+^p$ for every $t\geq 0$. Where $\boldsymbol{R}_+^n$, $\boldsymbol{R}_+^m$, and $\boldsymbol{R}_+^p$, are the positive orthants of the real vector spaces $\boldsymbol{R}^n$, $\boldsymbol{R}^m$, and $\boldsymbol{R}^p$, respectively.

Lemma 1. The system (1) is (internally) positive if and only if A is a Metzler matrix, and $B\in\boldsymbol{R}_+^{n\times m}$, $C\in\boldsymbol{R}_+^{p\times n}$.

Proof. See, e.g., [4].

Lemma 2. The system (1) is asymptotically stable if and only if all the eigenvalues of its system matrix A have negative real parts.

Proof. See, e.g., [4].

Let us consider the following linear time-discrete system

$$x(i+1)=Ax(i)+Bu(i),\quad x(0)=x_0, \tag{2a}$$

$$y(i)=Cx(i),\quad i=1,2,3,\ldots \tag{2b}$$

where $x(i)\in\boldsymbol{R}^n$, $y(i)\in\boldsymbol{R}^p$, and $u(i)\in\boldsymbol{R}^m$ are the state, output, and input vectors, respectively, and $A\in\boldsymbol{R}^{n\times n}$, $B\in\boldsymbol{R}^{n\times m}$, $C\in\boldsymbol{R}^{p\times n}$.

Definition 4. [4] The system (2) is called (internally) positive if for any $x_0\in\boldsymbol{R}_+^n$ and every input $u\in\boldsymbol{R}_+^m$ one has $x\in\boldsymbol{R}_+^n$ and $y\in\boldsymbol{R}_+^p$ for every $i\geq 0$.

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Lemma 3. The system (1) is (internally) positive if and only if A, B and C are nonnegative matrices, i.e., $A \in \boldsymbol{R}_{+}^{n\times n}$, $B \in \boldsymbol{R}_{+}^{n\times m}$, $C \in \boldsymbol{R}_{+}^{p\times n}$.

Proof. See, e.g., [4].

Lemma 4. The system (2) is asymptotically stable if and only if all the eigenvalues of its system matrix A have moduli less than 1.

Proof. See, e.g., [4].

2.2. Linear matrix inequalities. The set of $n \times n$ symmetric matrices is denoted by $\boldsymbol{S}^n$. The set of $n\times n$ diagonal matrices is denoted by $\boldsymbol{S}_d^n$. Obviously, $\boldsymbol{S}_d^n \subset \boldsymbol{S}^n$.

We say that $Q \in \boldsymbol{S}^n$ is positive definite (positive semidefinite) if its quadratic form is positive, i.e., $\forall x \in \boldsymbol{R}^n$, $x \neq 0$, $x^{\mathrm{T}}Qx > 0$ (nonnegative, i.e., $\forall x \in \boldsymbol{R}^n$ $x^{\mathrm{T}}Qx \geq 0$). We denote this fact by $Q \succ 0$ ($Q \succeq 0$). The negative definiteness (negative semidefiniteness) is defined in a similar way.

Let $\boldsymbol{V}$ denote a finite-dimensional real vector space with an associated inner product $\langle\cdot,\cdot\rangle_{\boldsymbol{V}}$, and let $\langle yz\rangle_{\boldsymbol{V}} := \mathbf{tr}(y^{\mathrm{T}}z)$, where y^{T} denotes the transpose of the vector y. In a similar way we define $\langle Y,Z\rangle_{\boldsymbol{S}^n} := \mathbf{tr}(Y^{\mathrm{T}}Z)$. By definition $Y^{\mathrm{T}} = Y$ for any symmetric matrix $Y \in \boldsymbol{S}^n$, thus one obtains $\langle Y,Z\rangle_{\boldsymbol{S}^n} = \mathbf{tr}(YZ)$.

Definition 5. [26] A linear matrix inequality (LMI) in the variable x is an inequality of the form

$$\mathcal{F}(x) + F_0 \succeq 0, \tag{3}$$

where the variable x takes values in the real vector space $\boldsymbol{V}$, the mapping $\mathcal{F} : \boldsymbol{V} \to \boldsymbol{S}^n$ is linear, and $F_0 \in \boldsymbol{S}^n$.

Remark 1. A set of LMIs can always be converted to one LMI, that is, the list of LMIs $\mathcal{F}_1(x_1) + F_1 \succeq 0$, $\mathcal{F}_2(x_2) + F_2 \succeq 0$, …, $\mathcal{F}_q(x_q) + F_q \succeq 0$ in the variables $x_1 \in \boldsymbol{V}_1$, $x_2 \in \boldsymbol{V}_2$, …, $x_q \in \boldsymbol{V}_q$ are equivalent to one LMI $\mathcal{F}(x) + F \succeq 0$, where

$$x = (x_1, x_2, \ldots, x_q) \in \boldsymbol{V}_1 \times \boldsymbol{V}_2 \times \ldots \times \boldsymbol{V}_q,$$

F is a block-diagonal matrix

$$F = \mathbf{diag}\left[F_1, F_2, \ldots, F_q\right],$$

and the function $\mathcal{F} : \boldsymbol{V}_1 \times \boldsymbol{V}_2 \times \ldots \times \boldsymbol{V}_q \to \boldsymbol{S}^{n_1+n_2+\ldots+n_q}$ is given by

$$\mathcal{F}(x) = \mathbf{diag}\left[\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_q\right].$$

Definition 6. For the linear mapping $\mathcal{F} : \boldsymbol{V} \to \boldsymbol{S}^n$ we define its adjoint mapping $\mathcal{F}^* : \boldsymbol{S} \to \boldsymbol{V}$ such that for all $x \in \boldsymbol{V}$ and $Z \in \boldsymbol{S}$, $\langle \mathcal{F}(x), Z\rangle_{\boldsymbol{S}} = \langle x, \mathcal{F}^*(Z)\rangle_{\boldsymbol{V}}$.

In our considerations we discriminate the following three kinds of feasibility:

1. *Strict feasibility:* $\exists x \in \boldsymbol{V}$ with $\mathcal{F}(x) + F_0 \succ 0$.
2. *Nonzero feasibility:* $\exists x \in \boldsymbol{V}$ with $\mathcal{F}(x) + F_0 \succeq\neq 0$ (i.e., positive semidefinite and nonzero).
3. *Feasibility:* $\exists x \in \boldsymbol{V}$ with $\mathcal{F}(x) + F_0 \succeq 0$.

Lemma 5. Suppose that F and Z are symmetric matrices of the same size, and that $F \succ 0$, $Z \succneqq 0$. Then

$$\mathbf{tr}(FZ) > 0 \ .$$

Proof. See, e.g., [27].

3. Alternative stability criterion for continuous time positive systems

Before proceeding to the main result of the section we provide an auxiliary lemma. We shall make use of some results from Section II.

Lemma 6. Suppose that A is a Metzler matrix, i.e., $A \in \boldsymbol{M}^n$. The matrix A is a Hurwitz matrix if and only if the following LMIs are feasible with respect to the diagonal matrix variable X (i.e., $X_{ij} = 0$ for $i,j = 1,2,\ldots,n \quad i \neq j$):

$$\begin{bmatrix} -(A^{\mathrm{T}}X + XA) & 0 \\ 0 & X \end{bmatrix} \succ 0. \tag{4}$$

Proof. See, e.g., [3,24].

The alternative stability criterion for positive continuous-time systems is as follows.

Proposition 1. Suppose that A is a Metzler matrix, i.e., $A \in \boldsymbol{M}^n$.

The matrix $A \in \boldsymbol{M}^n$ is Hurwitz if and only if the following LMIs are infeasible with respect to the matrix variable Y

$$Y = Y^{\mathrm{T}} \succneqq 0, \tag{5a}$$

$$I \circ [AY] \succeq 0, \tag{5b}$$

where I stands for identity matrix of appropriate dimensions and the symbol $\circ$ denotes the Hadamard product of two matrices (i.e., componentwise multiplication). In other words, $A \in \boldsymbol{M}^n$ has at least one eigenvalue with nonegative real part if and only if LMIs (5a–5b) are feasible.

Proof. Let us note that with $Z_1 := Y$ LMIs (5a–5b) can be rewritten as follows:

$$Z = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} \succneqq 0, \tag{6a}$$

$$Z_2 = I \circ \left[AZ_1 + Z_1A^{\mathrm{T}}\right], \tag{6b}$$

Let us note that (6a–6b) and (4) and contradict each other:

$$\begin{aligned} 0 &< \left\langle \begin{bmatrix} -(A^{\mathrm{T}}X + XA & 0 \\ 0 & X \end{bmatrix}, \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} \right\rangle_{\boldsymbol{S}^n \times \boldsymbol{S}_d^n} \\ &= \left\langle X, Z_2 - I \circ [AZ_1 + Z_1A^{\mathrm{T}}]\right\rangle_{\boldsymbol{S}_d^n} \\ &= 0, \end{aligned}$$

where the inequality follows by virtue of Lemma .5. Thus (4) and (6) cannot hold simultaneously.

Now, it remains to show that infeasibility of (4) implies feasibility of (6). To this end consider the set

$$\mathcal{C} = \left\{ U \in \boldsymbol{S}^n \times \boldsymbol{S}_d^n : \begin{bmatrix} -(A^{\mathrm{T}}X + XA & 0 \\ 0 & X \end{bmatrix} + U \succ 0 \right.$$
$$\left. \text{for some } \; X \in \boldsymbol{S}_d^n \right\}.$$

Suppose that (4) does not hold, i.e., $0 \notin \mathcal{C}$. Since $\mathcal{C}$ is open, nonempty and convex, there exists a hyperplane separating 0 from $\mathcal{C}$, i.e., there exists a nonzero $Z \in \boldsymbol{S}^n \times \boldsymbol{S}_d^n$ that satisfies

$$\langle 0, Z \rangle_{\boldsymbol{S}^n \times \boldsymbol{S}_d^n} \leq \langle U, Z \rangle_{\boldsymbol{S}^n \times \boldsymbol{S}_d^n}$$

for all $U \in \mathcal{C}$. This means that Z must satisfy $Z \neq 0$ and

$$0 \leqslant \left\langle Q - \begin{bmatrix} -(A^{\mathrm{T}}X + XA & 0 \\ 0 & X \end{bmatrix}, \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} \right\rangle_{\boldsymbol{S}^n \times \boldsymbol{S}_d^n}$$
$$= -\left\langle X, Z_2 - I \circ \left[AZ_1 + Z_1 A^{\mathrm{T}} \right] \right\rangle_{\boldsymbol{S}_d^n} +$$
$$+ \left\langle Q, \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} \right\rangle_{\boldsymbol{S}^n \times \boldsymbol{S}_d^n}$$

for all $Q \succ 0$ and all $X \in \boldsymbol{S}_d^n$. Let us note that $-\left\langle X, Z_2 - I \circ \left[AZ_1 + Z_1 A^{\mathrm{T}} \right] \right\rangle_{\boldsymbol{S}_d^n}$ is unbounded below with respect to the variable X if $Z_2 - I \circ \left[AZ_1 + Z_1 A^{\mathrm{T}} \right] \neq 0$, and it is equal to zero if $Z_2 - I \circ \left[AZ_1 + Z_1 A^{\mathrm{T}} \right] = 0$. Since Z defines a separating hyperplane, it must satisfy $Z_2 - I \circ \left[AZ_1 + Z_1 A^{\mathrm{T}} \right] = 0$. We also have that

$$\left\langle Q, \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} \right\rangle_{\boldsymbol{S}^n \times \boldsymbol{S}_d^n}$$

is unbounded below with respect to the variable $Q \succ 0$ if $Z \not\succeq 0$. Thus we obtain the second condition: $Z \succeq 0$. In summery, Z satisfies

$$Z = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} \gneqq 0, \quad Z_2 - I \circ [AZ_1 + Z_1 A^T] = 0.$$

Example 1. Let us consider the following continuous-time system

$$\dot{x}(t) = \underbrace{\begin{bmatrix} -1.1 & 0.1 & 0.6 & 0.4 \\ 0.3 & -1.0 & 1.5 & 1.2 \\ 3.0 & 1.3 & -0.2 & 0.3 \\ 0.1 & 0.4 & 0.5 & -1.3 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 1.4 \\ 3.3 \\ 3.4 \\ 0.3 \end{bmatrix}}_{B} u(t), \tag{7}$$
$$y(t) = \underbrace{\begin{bmatrix} 1.4 & 4.1 & 2.1 & 0.4 \end{bmatrix}}_{C} x(t),$$

Obviously, A is a Metzler matrix and it has the following eigenvalues $\lambda_1 = 1.6650$, $\lambda_2 = -2.4407$, $\lambda_3 = -1.1648$, $\lambda_4 = -1.6595$. Since $\lambda_1 > 0$ the matrix A is not Hurwitz. In fact, for the matrix

$$Y = \begin{bmatrix} 0.8053 & 0.8476 & 1.7255 & 0.5555 \\ 0.8476 & 3.4660 & 2.5872 & 1.0567 \\ 1.7255 & 2.5872 & 9.4887 & 2.3628 \\ 0.5555 & 1.0567 & 2.3628 & 0.9981 \end{bmatrix} \gneqq 0,$$

the conditions (5) are satisfied. One can also verify that for $Z_1 := Y$ and

$$Z_2 = \mathbf{diag}\,[\, 0.9128 \;\; 3.8741 \;\; 14.7020 \;\; 0.7242 \,] \succ 0,$$

the conditions (6) are satisfied.

4. Alternative stability criterion for discrete time positive systems

Before proceeding to the main result of the section we provide an auxiliary lemma. Once again, we shall make use of some results from Section 2.

Lemma 7. Suppose that A is a nonnegative matrix, i.e., $A \in \boldsymbol{R}_+^{n\times n}$. The matrix A is a Schur matrix if and only if the following LMIs are feasible with respect to the diagonal matrix variable X (i.e., $X_{ij} = 0$ for $i, j = 1, 2, \ldots, n \quad i \neq j$).

$$\begin{bmatrix} X - A^{\mathrm{T}}XA & 0 \\ 0 & X \end{bmatrix} \succ 0, \tag{8}$$

Proof. See, e.g., [3].

Proposition 2. Suppose that A is a nonnegative matrix, i.e., $A \in \boldsymbol{R}_+^{n\times n}$. The matrix $A \in \boldsymbol{R}_+^{n\times n}$ is a Schur matrix if and only if the following LMIs are infeasible with respect to the matrix variable Y

$$Y = Y^{\mathrm{T}} \gneqq 0, \tag{9a}$$

$$I \circ \left[AYA^{\mathrm{T}} - Y \right] \succeq 0, \tag{9b}$$

where I stands for identity matrix of appropriate dimensions and the symbol $\circ$ denotes the Hadamard product of two matrices (i.e., componentwise multiplication). In other words, $A \in \boldsymbol{R}_+^{n\times n}$ has at least one eigenvalue with modulus greater or equal to 1 if and only if LMIs (9a–9b) are feasible.

Proof. Let us note that with $Z_1 = Y$ LMIs (9a–9b) can be rewritten as follows.

$$Z = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} \gneqq 0, \tag{10a}$$

$$Z_2 = I \circ \left[AZ_1 A^{\mathrm{T}} - Z_1 \right], \tag{10b}$$

Now the proof follows in the same vein as that of Proposition 1.

Example 2. Let us consider the following discrete-time system

$$x(i+1) = \underbrace{\begin{bmatrix} 0.9 & 0.1 & 0.7 & 0.1 \\ 0.3 & 0.1 & 0.1 & 0.2 \\ 0.3 & 0.5 & 0.2 & 0.1 \\ 0.1 & 0.4 & 0.1 & 0.3 \end{bmatrix}}_{A} x(i) + \underbrace{\begin{bmatrix} 1.4 \\ 3.3 \\ 3.4 \\ 0.3 \end{bmatrix}}_{B} u(i),$$

$$y(i) = \underbrace{\begin{bmatrix} 1.4 & 4.1 & 2.1 & 0.4 \end{bmatrix}}_{C} x(i), \quad i = 1, 2, 3... \tag{11}$$

with $x(0) = x_0$. Obviously, A is a nonnegative matrix and it has the following eigenvalues $\lambda_1 = 1.2847$, $\lambda_2 = -0.0714 + 0.1716i$, $\lambda_3 = -0.0714 - 0.1716i$, $\lambda_4 = 0.3582$. Since $|\lambda_1| > 1$ the matrix A is not a Schur matrix. In fact, for the matrix

$$Y = \begin{bmatrix} 2.4557 & -0.0761 & -0.9800 & -0.0544 \\ -0.0761 & 1.5545 & 0.0737 & 0.0620 \\ -0.9800 & 0.0737 & 1.7387 & 0.0728 \\ -0.0544 & 0.0620 & 0.0728 & 1.6996 \end{bmatrix} \gneq 0,$$

the conditions (9) are satisfied. One can also verify that for $Z_1 := Y$ and

$$Z_2 = \mathbf{diag} \begin{bmatrix} 0.8186 & 1.2956 & 1.1623 & 1.2597 \end{bmatrix} \succ 0,$$

the conditions (10) are satisfied.

5. Generalized approach to the alternative stability criteria for positive systems

This paper has been inspired by [26]. In particular, [26] provides us with the following theorem.

Theorem 1. [26] Exactly one of the following statements is true.

1. $\exists x \in \boldsymbol{V}$ with $\mathcal{F}(x) + F_0 \succ 0$.
2. $\exists Z \in \boldsymbol{S}$ with $Z \gneq 0$, $\mathcal{F}^*(Z) = 0$, $\langle F_0, Z \rangle_{\boldsymbol{S}} \leq 0$.

Proof. See [26] and the references therein.

Since exactly one of the statements in the theorem is true they are called *strong alternatives* [26].

Remark 1. Geometrical interpretation of Theorem 1 is as follows. Statement 1) says that there exists nonempty intersection of the image of the linear map $\mathcal{F}$ translated from the origin of the space of symmetric matrices by F_0 and the interior of the cone of positive semidefinite matrices. Statement 2) says that there exists nonempty intersection of the null space over $\mathcal{F}^*$ (that is the orthogonal complement of the image of $\mathcal{F}$, since $\mathcal{N}(\mathcal{F}^*) = \mathcal{R}(\mathcal{F})^{\perp}$) with the cone of positive semidefinite matrices and with the halfspace $\langle F_0, Z \rangle_{\boldsymbol{S}} \leq 0$, and that this intersection is not equal to $\{0\}$.

Remark 2. The proof of Proposition 1 makes use of the ideas presented in the proof of Theorem 1 in [26].

Now let us return to Proposition 1 and Proposition 2.

Let $\mathcal{F}_{\mathbf{c}} : \boldsymbol{S}_d^n \to \boldsymbol{S}^n \times \boldsymbol{S}_d^n$, and $\mathcal{F}_{\mathbf{d}} : \boldsymbol{S}_d^n \to \boldsymbol{S}^n \times \boldsymbol{S}_d^n$ be defined in the following way

$$\mathcal{F}_{\mathbf{c}}(X) = \begin{bmatrix} -(A^T X + XA) & 0 \\ 0 & X \end{bmatrix},$$

$$\mathcal{F}_{\mathbf{d}}(X) = \begin{bmatrix} X - A^T XA) & 0 \\ 0 & X \end{bmatrix}.$$

Then it is easily verified that $\mathcal{F}_{\mathbf{c}}^* : \boldsymbol{S}^n \times \boldsymbol{S}_d^n \to \boldsymbol{S}_d^n$ is given by $\mathcal{F}_c^*(Z) = -I \circ (AZ_1) - I \circ (Z_1 A^{\mathrm{T}}) + Z_2$ and $\mathcal{F}_{\mathbf{d}}^* : \boldsymbol{S}^n \times \boldsymbol{S}_d^n \to \boldsymbol{S}_d^n$ is given by $\mathcal{F}_d^*(Z) = Z_2 - I \circ [AZ_1 A^{\mathrm{T}} - Z_1]$, where

$$Z = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}.$$

Now, the proof of Proposition 1 follows immediately from Theorem 1 and Lemma 6, and the proof of Proposition 2 from Theorem 1 and Lemma 7, respectively.

6. Concluding remarks and open problems

The stability problem for continuous time and discrete time positive systems have been discussed. An alternative formulation of stability criteria for positive systems have been proposed. The results have been illustrated with numerical examples. It should be stressed that appropriate software is necessary to apply the proposed results. All numerical examples provided in the paper have been solved using MATLAB® environment together with SEDUMI® solver and YALMIP® parser. More details on the computational aspects can be found in [28–31]. The generalization of the presented results onto weakly positive systems [4] remains an open problem.

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