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Dual hesitant pythagorean fuzzy Hamacher aggregation operators in multiple attribute decision making

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In this paper, we investigate the multiple attribute decision making (MADM) problem based on the Hamacher aggregation operators with dual Pythagorean hesitant fuzzy information. Then, motivated by the ideal of Hamacher operation, we have developed some Hamacher aggregation operators for aggregating dual hesitant Pythagorean fuzzy information. The prominent characteristic of these proposed operators are studied. Then, we have utilized these operators to develop some approaches to solve the dual hesitant Pythagorean fuzzy multiple attribute decision making problems. Finally, a practical example for supplier selection in supply chain management is given to verify the developed approach and to demonstrate its practicality and effectiveness.

Key words: multiple attribute decision making (MADM); dual Pythagorean hesitant fuzzy values; dual hesitant Pythagorean fuzzy Hamacher hybrid average (DHPFHHA) operator; dual hesitant Pythagorean fuzzy Hamacher hybrid geometric (DHPFHGG) operator; power aggregation operators.

1. Introduction

Atanassov [1,2] introduced the concept of intuitionistic fuzzy set (IFS) characterized by a membership function and a non-membership function, which is a generalization of the concept of fuzzy set [3] whose basic component is only a membership function. Xu [4] developed the intuitionistic fuzzy weighted averaging (IFWA) operator, intuitionistic fuzzy ordered weighted averaging (IFOWA) operator and the intuitionistic fuzzy hybrid aggregation (IFHA) operator. Xu [5] developed some geometric aggregation operators, such as the intuitionistic fuzzy weighted geometric (IFWG) operator, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator, and the intuitionistic fuzzy hybrid geometric (IFHG) operator and gave an application of the IFHG

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operator to multiple attribute group decision making with intuitionistic fuzzy information. Xu and Yager [6] investigated the dynamic intuitionistic fuzzy multiple attribute decision making problems and developed some aggregation operators such as the dynamic intuitionistic fuzzy weighted averaging (DIFWA) operator and uncertain dynamic intuitionistic fuzzy weighted averaging (UDIFWA) operator to aggregate dynamic or uncertain dynamic intuitionistic fuzzy information. Wei [7] proposed some dynamic geometric aggregation operators such as the dynamic intuitionistic fuzzy weighted geometric (DIFWG) operator and uncertain dynamic intuitionistic fuzzy weighted geometric (UDIFWG) operator to aggregate dynamic or uncertain dynamic intuitionistic fuzzy information. Wei [8] proposed two new aggregation operators: induced intuitionistic fuzzy ordered weighted geometric (I-IFOWG) operator and induced interval-valued intuitionistic fuzzy ordered weighted geometric (I-IIFOWG) operator. Wei and Zhao [9] developed two new aggregation operators: induced intuitionistic fuzzy correlated averaging (I-IFCA) operator and induced intuitionistic fuzzy correlated geometric (I-IFCG) operator. Yu et al. [10] proposed some intuitionistic fuzzy aggregation operators such as the intuitionistic fuzzy prioritized weighted average (IFPWA) operator, the intuitionistic fuzzy prioritized weighted geometric (IFPWG) operator. Xu [11] developed a series of operators for aggregating intuitionistic fuzzy numbers and established various properties of these power aggregation operators. Xu and Chen [12] proposed an aggregation technique called the intuitionistic fuzzy Bonferroni mean for aggregating intuitionistic fuzzy information. Xu and Xia [13] studied the induced generalized aggregation operators under intuitionistic fuzzy environments. The intuitionistic fuzzy set has received more and more attention since its appearance [14-28].

More recently, Pythagorean fuzzy set (PFS) [29-30] has emerged as an effective tool for depicting uncertainty of the MADM problems. The PFS is also characterized by the membership degree and the non-membership degree, whose sum of squares is less than or equal to 1, the PFS is more general than the IFS. In some cases, the PFS can solve the problems that the IFS cannot, for example, if a DM gives the membership degree and the non-membership degree as 0.8 and 0.6, respectively, then it is only valid for the PFS. In other words, all the intuitionistic fuzzy degrees are a part of the Pythagorean fuzzy degrees, which indicates that the PFS is more powerful to handle the uncertain problems. Zhang and Xu [31] provided the detailed mathematical expression for PFS and introduced the concept of Pythagorean fuzzy number (PFN). Meanwhile, they also developed a Pythagorean fuzzy TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) for handling the MCDM problem within PFNs. Peng and Yang [32] proposed the division and subtraction operations for PFNs, and also developed a Pythagorean fuzzy superiority and inferiority ranking method to solve multicriteria group decision making problem with PFNs. Afterwards, Beliakov and James [33] focused on how the notion of "averaging" should be treated in the case of PFNs and how to ensure that the averaging aggregation functions produce outputs consistent with the case of ordinary fuzzy numbers. Reformat and Yager [34] applied the PFNs in handling the collaborative-based recommender system.

In this paper, we introduce dual hesitant Pythagorean fuzzy set (DHPFS), which is a new extension of PFS and dual hesitant fuzzy set (DHF) [35]. It's clear that the DHPFSs consist of two parts, that is, the membership degrees function and the non-membership degrees function, supporting a more exemplary and flexible access to assign values for each element in the domain, and we have to handle two kinds of degrees in this situation. For example, in a multiple attribute decision-making problem, some decision makers consider as possible values for the membership degree of into the set a few different values 0.4, 0.5, and 0.6, and for the non-membership degrees 0.1, 0.2 and 0.3 replacing just one number or a tuple. So, the certainty and uncertainty on the possible values are somehow limited, respectively, which can reflect the original information given by the decision makers as much as possible. Utilizing DHPFSs can take much more information into account, the more values we obtain from the decision makers, the greater epistemic certainty we have, and thus, compared to the existing sets, DHPFS can be regarded as a more comprehensive set, which supports a more flexible approach when the decision makers provide their judgments.

Hamacher t-conorm and t-norm, which are the generalization of algebraic and Einstein t-conorm and t-norm [36], are more general and more flexible. There is important significance to research aggregation operators based on Hamacher operations and their application to MADM problems. However, all the above approaches are unsuitable to aggregate these dual hesitant Pythagorean fuzzy numbers on the basis of the Hamacher operations [37]. Thus, based on the Hamacher operations, how to aggregate these dual hesitant Pythagorean fuzzy numbers is an interesting topic. To solve this issue, in this paper, we shall develop some dual hesitant Pythagorean fuzzy Hamacher aggregation operators on the basis of the traditional Hamacher operations [37-42]. In order to do so, the remainder of this paper is set out as follows. In the next section, we introduce some basic concepts related to Pythagorean fuzzy set, dual hesitant Pythagorean fuzzy set and their operational laws. In Section 3, we shall propose some dual hesitant Pythagorean fuzzy Hamacher aggregation operators. In Section 4, we shall propose some dual hesitant Pythagorean fuzzy Hamacher power aggregation operators. In Section 5, based on these operators, we shall propose some models for multiple attribute decision making problems with dual hesitant Pythagorean fuzzy information. In Section 6, we present a numerical example for supplier selection in supply chain management with dual hesitant Pythagorean fuzzy information in order to illustrate the method proposed in this paper. Section 7 concludes the paper with some remarks.

2. Preliminaries

2.1. Pythagorean fuzzy set

The basic concepts of PFSs [29-30] are briefly reviewed in this section. Afterwards, novel score and accuracy functions for PFNs are proposed. Furthermore, a new comparison method for PFNs is developed.

Definition 1 [29-30] Let X be a fix set. A PFS is an object having the form

$$P = \{ \langle x, (\mu_P(x), \nu_P(x)) \rangle \mid x \in X \} \quad (1)$$

where $\mu_P: X \rightarrow [0, 1]$ the function defines the degree of membership and the function $\nu_P: X \rightarrow [0, 1]$ defines the degree of non-membership of the element $x \in X$ to P , respectively, and, for every $x \in X$, it holds that

$$(\mu_P(x))^2 + (\nu_P(x))^2 \leq 1. \quad (2)$$

Definition 2 [31] Let $\tilde{a}_1 = (\mu_1, \nu_1)$, $\tilde{a}_2 = (\mu_2, \nu_2)$, and $\tilde{a} = (\mu, \nu)$ be three Pythagorean fuzzy numbers, and some basic operations on them are defined as follows:

$$(1) \tilde{a}_1 \oplus \tilde{a}_2 = \left(\sqrt{(\mu_1)^2 + (\mu_2)^2 - (\mu_1)^2(\mu_2)^2}, \nu_1\nu_2 \right);$$

$$(2) \tilde{a}_1 \otimes \tilde{a}_2 = \left(\mu_1\mu_2, \sqrt{(\nu_1)^2 + (\nu_2)^2 - (\nu_1)^2(\nu_2)^2} \right);$$

$$(3) \lambda \tilde{a} = \left(\sqrt{1 - (1 - \mu^2)^\lambda}, \nu^\lambda \right), \lambda > 0;$$

$$(4) (\tilde{a})^\lambda = \left(\mu^\lambda \sqrt{1 - (1 - \nu^2)^\lambda}, \nu \right), \lambda > 0;$$

$$(5) \tilde{a}^c = (\nu, \mu).$$

2.2. Dual hesitant Pythagorean fuzzy set

In this section, we introduce dual hesitant Pythagorean fuzzy set (DHPFS), which is a new extension of PFS and dual hesitant fuzzy set [35]. It is clear that the DHPFSs consist of two parts, that is, the membership hesitancy function and the non-membership hesitancy function, supporting a more exemplary and flexible access to assign values for each element in the domain, and we have to handle two kinds of hesitancy in this situation.

Definition 3 Let X be a fixed set, then a dual hesitant Pythagorean fuzzy set (DHPFS) on X is described as:

$$D = (\langle x, (h_P(x), g_P(x)) \rangle \mid x \in X) \quad (3)$$

in which $h_P(x)$ and $g_P(x)$ are two sets of some values in $[0, 1]$, denoting the possible membership degrees and non-membership degrees of the element $x \in X$ to the set D respectively, with the conditions:

$$\gamma^2 + \eta^2 \leq 1$$

where $\gamma \in h_P(x)$, $\eta \in g_P(x)$, for all $x \in X$. For convenience, the pair $d(x) = (h_P(x), g_P(x))$ is called a dual hesitant Pythagorean fuzzy number (DHPFN) denoted by $d = (h, g)$, with the conditions: $\gamma \in h$, $\eta \in g$, $0 \leq \gamma$, $\eta \leq 1$, $0 \leq \gamma^2 + \eta^2 \leq 1$.

To compare the DHPFNs, in the following, we shall give the following comparison laws:

Definition 4 Let $d = (h, g)$ be a DHPFNs, $s(d) = \frac{1}{2} \left(1 + \frac{1}{\#h} \sum_{\gamma \in h} \gamma^2 - \frac{1}{\#g} \sum_{\eta \in g} \eta^2 \right)$ the score function of d , and $p(d) = \frac{1}{\#h} \sum_{\gamma \in h} \gamma^2 + \frac{1}{\#g} \sum_{\eta \in g} \eta^2$ the accuracy function of d , where $\#h$ and $\#g$ are the numbers of the elements in h and g respectively, then, let $d_i = (h_i, g_i)$ ($i = 1, 2$) be any two DHPFNs, we have the following comparison laws:

- If $s(d_1) > s(d_2)$, then d_1 is superior to d_2 , denoted by $d_1 \succ d_2$;
- If $s(d_1) = s(d_2)$, then
 - (1) If $p(d_1) = p(d_2)$, then d_1 is equivalent to d_2 , denoted by $d_1 \sim d_2$;
 - (2) If $p(d_1) > p(d_2)$, then d_1 is superior to d_2 , denoted by $d_1 \succ d_2$.

Example 1 Let $d_1 = \{ \{ 0.3, 0.4 \}, \{ 0.6 \} \}$, $d_2 = \{ \{ 0.4, 0.5 \}, \{ 0.3, 0.4 \} \}$ by Definition 4, we can get:

$$s(d_1) = \frac{1}{2} \left(1 + \frac{1}{2} (0.3^2 + 0.4^2) - 0.6^2 \right) = 0.3825$$

$$s(d_2) = \frac{1}{2} \left(1 + \frac{1}{2} (0.4^2 + 0.5^2) - \frac{1}{2} (0.3^2 + 0.4^2) \right) = 0.5400$$

Thus, $s(d_2) > s(d_1)$, so $d_2 \succ d_1$. Then, we define some **new** operations on the DHPFNs d, d_1 and d_2 :

- (1) $d^\lambda = \cup_{\gamma \in h, \eta \in g} \left\{ \{ \gamma^\lambda \}, \left\{ \sqrt{1 - (1 - \eta^2)^\lambda} \right\} \right\}, \lambda > 0$;
- (2) $\lambda d = \cup_{\gamma \in h, \eta \in g} \left\{ \left\{ \sqrt{1 - (1 - \gamma^2)^\lambda} \right\}, \{ \eta^\lambda \} \right\}, \lambda > 0$;
- (3) $d_1 \oplus d_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{ \left\{ \sqrt{(\gamma_1)^2 + (\gamma_2)^2 - (\gamma_1)^2 (\gamma_2)^2} \right\}, \{ \eta_1 \eta_2 \} \right\}$;
- (4) $d_1 \otimes d_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{ \{ \gamma_1 \gamma_2 \}, \left\{ \sqrt{(\eta_1)^2 + (\eta_2)^2 - (\eta_1)^2 (\eta_2)^2} \right\} \right\}$.

2.3. Hamacher operations of dual hesitant Pythagorean fuzzy set

Based on the traditional Hamacher operations [36], in the following, we shall define the Hamacher operations on the DHPFNs d, d_1 and d_2 .

$$(1) \quad d^\lambda = \cup_{\gamma \in h, \eta \in g} \left\{ \left\{ \frac{\sqrt{\gamma}(\gamma)^\lambda}{\sqrt{(1+(\gamma-1)(1-(\gamma)^2))^\lambda + (\gamma-1)(\gamma)^{2\lambda}}} \right\} \right\},$$

$$(2) \quad \lambda d = \cup_{\gamma \in h, \eta \in g} \left\{ \left\{ \sqrt{\frac{(1+(\gamma-1)(\eta_1)^2)^\lambda - (1-(\eta_1)^2)^\lambda}{(1+(\gamma-1)(\eta_1)^2)^\lambda + (\gamma-1)(1-(\eta_1)^2)^\lambda}} \right\} \right\}, \lambda > 0;$$

$$\left\{ \left\{ \sqrt{\frac{(1+(\gamma-1)(\eta_1)^2)^\lambda - (1-(\eta_1)^2)^\lambda}{(1+(\gamma-1)(\eta_1)^2)^\lambda + (\gamma-1)(1-(\eta_1)^2)^\lambda}} \right\} \right\},$$

$$\left\{ \frac{\sqrt{\gamma}(\eta_1)^\lambda}{\sqrt{(1+(\gamma-1)(1-(\eta_1)^2))^\lambda + (\gamma-1)(\eta_1)^{2\lambda}}} \right\}, \lambda > 0$$

$$(3) \quad d_1 \oplus d_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{ \left\{ \sqrt{\frac{(\gamma_1)^2 + (\gamma_2)^2 - (\eta_1)^2(\eta_2)^2 - (1-\gamma)(\eta_1)^2(\eta_2)^2}{1 - (1-\gamma)(\eta_1)^2(\eta_2)^2}} \right\} \right\},$$

$$\left\{ \frac{\eta_1 \eta_2}{\sqrt{\gamma + (1-\gamma)((\eta_1)^2 + (\eta_2)^2 - (\eta_1)^2(\eta_2)^2)}} \right\}, \lambda > 0$$

$$(4) \quad d_1 \otimes d_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{ \left\{ \frac{\gamma_1 \gamma_2}{\sqrt{\gamma + (1-\gamma)((\eta_1)^2 + (\eta_2)^2 - (\eta_1)^2(\eta_2)^2)}} \right\} \right\},$$

$$\left\{ \sqrt{\frac{(\eta_1)^2 + (\eta_2)^2 - (\eta_1)^2(\eta_2)^2 - (1-\gamma)(\eta_1)^2(\eta_2)^2}{1 - (1-\gamma)(\eta_1)^2(\eta_2)^2}} \right\}, \lambda > 0$$

3. Dual hesitant Pythagorean fuzzy Hamacher aggregation operators

3.1. Dual hesitant Pythagorean fuzzy Hamacher arithmetic aggregation operators

In the following, we shall propose some dual hesitant Pythagorean fuzzy Hamacher arithmetic aggregation operator based on the Hamacher operations of DHPFNs.

Definition 5 Let $\tilde{d}_j (j = 1, 2, \dots, n)$ be a collection of DHPFNs, then we define the dual hesitant Pythagorean fuzzy Hamacher weighted average (DHPFHWA) operator as follows:

$$DHPFHWA_\omega(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) = \bigoplus_{j=1}^n (\omega_j \tilde{d}_j) \tag{4}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of $\tilde{d}_j (j = 1, 2, \dots, n)$, and $\omega_j > 0, \sum_{j=1}^n \omega_j = 1$.

Based on the operations of the dual hesitant Pythagorean fuzzy values described and mathematical induction method, we can drive the Theorem 1.

Theorem 1 Let $\tilde{d}_j (j = 1, 2, \dots, n)$ be a collection of DHPFNs, then their aggregated value by using the DHPFHWA operator is also a DHPFN, and

$$\begin{aligned}
 & \text{DHPFHWA}_{\omega}(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) \\
 &= \bigoplus_{j=1}^n (\omega_j \tilde{d}_j) \\
 &= \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \left\{ \sqrt{\frac{\prod_{j=1}^n \left(1 + (\gamma - 1)(\gamma_j)^2\right)^{\omega_j} - \prod_{j=1}^n \left(1 - (\gamma_j)^2\right)^{\omega_j}}{\prod_{j=1}^n \left(1 + (\gamma - 1)(\gamma_j)^2\right)^{\omega_j} + (\gamma - 1) \prod_{j=1}^n \left(1 - (\gamma_j)^2\right)^{\omega_j}}} \right\}, \right. \\
 & \quad \left. \left\{ \frac{\sqrt{\gamma} \prod_{j=1}^n (\eta_j)^{\omega_j}}{\sqrt{\prod_{j=1}^n \left(1 + (\gamma - 1)(1 - (\eta_j)^2)\right)^{\omega_j} + (\gamma - 1) \prod_{j=1}^n (\eta_j)^{2\omega_j}}} \right\} \right\}, \quad (5)
 \end{aligned}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of $\tilde{d}_j (j = 1, 2, \dots, n)$, and $\omega_j > 0$, $\sum_{j=1}^n \omega_j = 1$, $\gamma > 0$.

Now, we can discuss some special cases of the DHPFHWA operator with respect to the parameter γ .

- When $\gamma = 1$, DHPFHWA operator reduces to the dual hesitant Pythagorean fuzzy weighted average (DHPFWA) operator as follows:

$$\begin{aligned}
 & \text{DHPFWA}_{\omega}(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) \\
 &= \bigoplus_{j=1}^n (\omega_j \tilde{d}_j) \\
 &= \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \left\{ \sqrt{1 - \prod_{j=1}^n \left(1 - (\gamma_j)^2\right)^{\omega_j}} \right\}, \left\{ \prod_{j=1}^n (\eta_j)^{\omega_j} \right\} \right\} \quad (6)
 \end{aligned}$$

- When $\gamma = 2$, DHPFHWA operator reduces to the dual hesitant Pythagorean fuzzy Einstein weighted average (DHPFEWA) operator as follows:

$$\begin{aligned}
 & \text{DHPFEWA}_\omega (\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) \\
 &= \bigoplus_{j=1}^n (\omega_j \tilde{d}_j) \\
 &= \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \left\{ \frac{\prod_{j=1}^n (1 + (\gamma_j)^2)^{\omega_j} - \prod_{j=1}^n (1 - (\gamma_j)^2)^{\omega_j}}{\prod_{j=1}^n (1 + (\gamma_j)^2)^{\omega_j} + \prod_{j=1}^n (1 - (\gamma_j)^2)^{\omega_j}} \right\}, \right. \tag{7} \\
 & \left. \left\{ \frac{\sqrt{2} \prod_{j=1}^n (\eta_j)^{\omega_j}}{\sqrt{\prod_{j=1}^n (2 - (\eta_j)^2)^{\omega_j} + \prod_{j=1}^n (\eta_j)^{2\omega_j}}} \right\} \right\}
 \end{aligned}$$

Definition 6 Let $\tilde{d}_j (j = 1, 2, \dots, n)$ be a collection of DHPFNs, then we define the dual hesitant Pythagorean fuzzy Hamacher ordered weighted average (DHPFHOWA) operator as follows:

$$\begin{aligned}
 & \text{DHPFHOWA}_w (\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) = \bigoplus_{j=1}^n (w_j \tilde{d}_{\sigma(j)}) \\
 &= \cup_{\gamma_{\sigma(j)} \in h_{\sigma(j)}, \eta_{\sigma(j)} \in g_{\sigma(j)}} \left\{ \left\{ \frac{\prod_{j=1}^n (1 + (\gamma - 1) (\gamma_{\sigma(j)})^2)^{w_j} - \prod_{j=1}^n (1 - (\gamma_{\sigma(j)})^2)^{w_j}}{\prod_{j=1}^n (1 + (\gamma - 1) (\gamma_{\sigma(j)})^2)^{w_j} + (\gamma - 1) \prod_{j=1}^n (1 - (\gamma_{\sigma(j)})^2)^{w_j}} \right\}, \right. \\
 & \left. \left\{ \frac{\sqrt{\gamma} \prod_{j=1}^n (\gamma_{\sigma(j)})^{w_j}}{\sqrt{\prod_{j=1}^n (1 + (\gamma - 1) (1 - (\gamma_{\sigma(j)})^2))^{w_j} + (\gamma - 1) \prod_{j=1}^n (\gamma_{\sigma(j)})^{2w_j}}} \right\} \right\} \tag{8}
 \end{aligned}$$

where $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $1, 2, \dots, n$, such that $\tilde{d}_{\sigma(j-1)} \geq \tilde{d}_{\sigma(j)}$ for all $j = 2, \dots, n$, and $w = (w_1, w_2, \dots, w_n)^T$ is the aggregation-associated weight vector such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1, \gamma > 0$.

Now, we can discuss some special cases of the DHPFHOWA operator with respect to the parameter γ .

- When $\gamma = 1$, DHPFHOWA operator reduces to the dual hesitant Pythagorean fuzzy ordered weighted average (DHFOWA) operator as follows:

$$\begin{aligned} \text{DHPFOWA}_w(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) &= \bigoplus_{j=1}^n (w_j \tilde{d}_{\sigma(j)}) \\ &= \cup_{\gamma_{\sigma(j)} \in h_{\sigma(j)}, \eta_{\sigma(j)} \in g_{\sigma(j)}} \left\{ \left\{ \sqrt{1 - \prod_{j=1}^n (1 - (\gamma_{\sigma(j)})^2)^{w_j}} \right\}, \right. \\ &\quad \left. \left\{ \frac{\sqrt{2} \prod_{j=1}^n \gamma_j^{\omega_j}}{\sqrt{\prod_{j=1}^n (2 - (\gamma_j)^2)^{\omega_j} + \prod_{j=1}^n (\gamma_j)^{2\omega_j}}} \right\} \right\} \end{aligned} \tag{9}$$

- When $\gamma = 2$, DHPFHOWA operator reduces to the dual hesitant Pythagorean fuzzy Einstein ordered weighted average (DHPFEOWA) operator as follows:

$$\begin{aligned} \text{DHPFEOWA}_w(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) &= \bigoplus_{j=1}^n (w_j \tilde{d}_{\sigma(j)}) \\ &= \cup_{\gamma_{\sigma(j)} \in h_{\sigma(j)}, \eta_{\sigma(j)} \in g_{\sigma(j)}} \left\{ \left\{ \frac{\prod_{j=1}^n (1 + (\gamma_{\sigma(j)})^2)^{w_j} - \prod_{j=1}^n (1 - (\gamma_{\sigma(j)})^2)^{w_j}}{\prod_{j=1}^n (1 + (\gamma_{\sigma(j)})^2)^{w_j} + \prod_{j=1}^n (1 - (\gamma_{\sigma(j)})^2)^{w_j}} \right\}, \right. \\ &\quad \left. \left\{ \frac{\sqrt{2} \prod_{j=1}^n (\eta_{\sigma(j)})^{\omega_j}}{\sqrt{\prod_{j=1}^n (2 - (\eta_{\sigma(j)})^2)^{\omega_j} + \prod_{j=1}^n (\eta_{\sigma(j)})^{2\omega_j}}} \right\} \right\} \end{aligned} \tag{10}$$

From Definitions 5 and 6, we know that the DHPFHOWA operator weights the dual hesitant Pythagorean fuzzy argument itself, while the DHPFHOWA operator weights the ordered positions of the dual hesitant Pythagorean fuzzy arguments instead of weighting the arguments themselves. Therefore, weights represent different aspects in both the DHPFHOWA and DHPFHOWA operators. However, both the operators consider only one of them. To solve this drawback, in the following we shall propose a dual hesitant Pythagorean fuzzy Hamacher hybrid average (DHPFHHA) operator.

Definition 7 A dual hesitant Pythagorean fuzzy Hamacher hybrid average (DHPFHHA) operator is defined as follows:

$$\begin{aligned}
 DHPFHHA_{w,\omega}(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) &= \bigoplus_{j=1}^n \left(w_j \tilde{d}_{\sigma(j)} \right) \\
 &= \cup_{\gamma_{\sigma(j)} \in h_{\sigma(j)}, \dot{\eta}_{\sigma(j)} \in g_{\sigma(j)}} \left\{ \left\{ \frac{\prod_{j=1}^n \left(1 + (\gamma - 1) (\dot{\gamma}_{\sigma(j)})^2 \right)^{\omega_j} - \prod_{j=1}^n \left(1 - (\dot{\gamma}_{\sigma(j)})^2 \right)^{\omega_j}}{\sqrt{\prod_{j=1}^n \left(1 + (\gamma - 1) (\dot{\gamma}_{\sigma(j)})^2 \right)^{\omega_j} + (\gamma - 1) \prod_{j=1}^n \left(1 - (\dot{\gamma}_{\sigma(j)})^2 \right)^{\omega_j}}} \right\} \right. \\
 &\quad \left. \left\{ \frac{\sqrt{\gamma} \prod_{j=1}^n (\dot{\eta}_{\sigma(j)})^{\omega_j}}{\sqrt{\prod_{j=1}^n \left(1 + (\gamma - 1) \left(1 - (\dot{\eta}_{\sigma(j)})^2 \right) \right)^{\omega_j} + (\gamma - 1) \prod_{j=1}^n (\dot{\eta}_{\sigma(j)})^{2\omega_j}}} \right\} \right\} \tag{11}
 \end{aligned}$$

where $w = (w_1, w_2, \dots, w_n)$ is the associated weighting vector, with $w_j \in [0, 1]$, $\sum_{j=1}^n w_j = 1$, and $\dot{h}_{\sigma(j)}$ is the j -th largest element of the dual hesitant Pythagorean fuzzy arguments \tilde{d} ($\tilde{d} = n\omega_j \tilde{d}_j$, $j = 1, 2, \dots, n$), $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ is the weighting vector of dual hesitant Pythagorean fuzzy arguments \tilde{d}_j ($j = 1, 2, \dots, n$), with $\omega_i \in [0, 1]$, $\sum_{i=1}^n \omega_i = 1$, and n is the balancing coefficient, $\gamma > 0$. Especially, if $w = (1/n, 1/n, \dots, 1/n)^T$, then DHPFHHA is reduced to the dual hesitant Pythagorean fuzzy weighted average (DHPFWA) operator; if, then DHPFHHA is reduced to the dual hesitant Pythagorean fuzzy ordered weighted average (DHPFOWA) operator.

From Definition 7, we know that:

- (1) The DHPFHHA operator first weights the given arguments, and then reorders the weighted arguments in descending order and weights these ordered arguments by the DHPFHHA weights, and finally aggregates all the weighted arguments into a collective one.
- (2) The DHPFHHA operator generalizes both the DHPFHWa and DHPFHOWa operators, and reflects the importance degrees of both the given arguments and their ordered positions.

Now, we can discuss some special cases of the DHPFHHA operator with respect to the parameter γ .

- When $\gamma = 1$, DHPFHHA operator reduces to the hesitant Pythagorean fuzzy hybrid average (DHPFHHA) operator as follows:

$$\begin{aligned} \text{DHPFHHA}_{w,\omega}(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) &= \bigoplus_{j=1}^n (w_j \tilde{d}_{\sigma(j)}) \\ &= \cup_{\dot{\gamma}_{\sigma(j)} \in h_{\sigma(j)}, \dot{\eta}_{\sigma(j)} \in g_{\sigma(j)}} \left\{ \left\{ \sqrt{1 - \prod_{j=1}^n \left(1 - (\dot{\gamma}_{\sigma(j)})^2\right)^{w_j}} \right\}, \left\{ \prod_{j=1}^n (\dot{\eta}_{\sigma(j)})^{w_j} \right\} \right\} \end{aligned} \quad (12)$$

- When $\gamma = 2$, DHPFHHA operator reduces to the dual hesitant Pythagorean fuzzy Einstein hybrid average (HPFEHA) operator as follows:

$$\begin{aligned} \text{DHPFEHA}_{w,\omega}(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) &= \bigoplus_{j=1}^n (w_j \tilde{d}_{\sigma(j)}) \\ &= \cup_{\dot{\gamma}_{\sigma(j)} \in h_{\sigma(j)}, \dot{\eta}_{\sigma(j)} \in g_{\sigma(j)}} \left\{ \left\{ \frac{\prod_{j=1}^n \left(1 + (\dot{\gamma}_{\sigma(j)})^2\right)^{w_j} - \prod_{j=1}^n \left(1 - (\dot{\gamma}_{\sigma(j)})^2\right)^{w_j}}{\prod_{j=1}^n \left(1 + (\dot{\gamma}_{\sigma(j)})^2\right)^{w_j} + \prod_{j=1}^n \left(1 - (\dot{\gamma}_{\sigma(j)})^2\right)^{w_j}} \right\}, \right. \\ &\quad \left. \left\{ \frac{\sqrt{2} \prod_{j=1}^n (\dot{\eta}_{\sigma(j)})^{\omega_j}}{\sqrt{\prod_{j=1}^n \left(2 - (\dot{\eta}_{\sigma(j)})^2\right)^{\omega_j} + \prod_{j=1}^n (\dot{\eta}_{\sigma(j)})^{2\omega_j}}} \right\} \right\} \end{aligned} \quad (13)$$

3.2. Dual hesitant Pythagorean fuzzy Hamacher Geometric Aggregation Operators

Based on the dual hesitant Pythagorean fuzzy Hamacher arithmetic aggregation operators and the geometric mean, here we define some dual hesitant Pythagorean fuzzy Hamacher geometric aggregation operators:

Definition 8 Let $\tilde{d}_j (j = 1, 2, \dots, n)$ be a collection of DHPFNs, then we define the dual hesitant Pythagorean fuzzy Hamacher weighted geometric (DHPFHWG) operator as follows:

$$\text{DHPFHWG}_{\omega}(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) = \bigotimes_{j=1}^n (\tilde{d}_j)^{\omega_j} \quad (14)$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of $\tilde{d}_j (j = 1, 2, \dots, n)$, and $\omega_j > 0$, $\sum_{j=1}^n \omega_j = 1, \gamma > 0$.

Based on the operations of the dual hesitant Pythagorean fuzzy values described and mathematical induction methods, we can drive the Theorem 2.

Theorem 2 Let $\tilde{d}_j (j = 1, 2, \dots, n)$ be a collection of DHPFNs, then their aggregated value by using the DHPFHWG operator is also a DHPFN, and

$$\begin{aligned}
 & DHPFHWG_{\omega}(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) \\
 &= \bigotimes_{j=1}^n (\tilde{d}_j)^{\omega_j} \\
 &= \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \left\{ \frac{\sqrt{\gamma} \prod_{j=1}^n \gamma_j^{\omega_j}}{\sqrt{\prod_{j=1}^n (1 + (\gamma - 1)(1 - (\gamma_j)^2))^{\omega_j} + (\gamma - 1) \prod_{j=1}^n (\gamma_j)^{2\omega_j}}} \right\}, \right. \\
 & \quad \left. \left\{ \sqrt{\frac{\prod_{j=1}^n (1 + (\gamma - 1)(\eta_j)^2)^{\omega_j} - \prod_{j=1}^n (1 - (\eta_j)^2)^{\omega_j}}{\prod_{j=1}^n (1 + (\gamma - 1)(\eta_j)^2)^{\omega_j} + (\gamma - 1) \prod_{j=1}^n (1 - (\eta_j)^2)^{\omega_j}}} \right\} \right\}
 \end{aligned} \tag{15}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of $\tilde{d}_j (j = 1, 2, \dots, n)$, and $\omega_j > 0$, $\sum_{j=1}^n \omega_j = 1, \gamma > 0$.

Now, we can discuss some special cases of the DHPFHWG operator with respect to the parameter γ .

- When $\gamma = 1$, DHPFHWG operator reduces to the dual hesitant Pythagorean fuzzy weighted geometric (DHPFWG) operator as follows:

$$\begin{aligned}
 & DHPFWG_{\omega}(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) = \bigotimes_{j=1}^n (\tilde{d}_j)^{\omega_j} \\
 &= \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \left\{ \prod_{j=1}^n (\gamma_j)^{\omega_j} \right\}, \left\{ \sqrt{1 - \prod_{j=1}^n (1 - (\eta_j)^2)^{\omega_j}} \right\} \right\}
 \end{aligned} \tag{16}$$

- When $\gamma = 2$, DHPFHWG operator reduces to the dual hesitant Pythagorean fuzzy Einstein weighted geometric (DHPFEWG) operator as follows:

$$\begin{aligned} \text{DHPFEWG}_\omega \left(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n \right) &= \bigotimes_{j=1}^n \left(\tilde{d}_j \right)^{\omega_j} \\ &= \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \left\{ \frac{\sqrt{2} \prod_{j=1}^n \gamma_j^{\omega_j}}{\sqrt{\prod_{j=1}^n \left(2 - (\gamma_j)^2 \right)^{\omega_j} + \prod_{j=1}^n (\gamma_j)^{2\omega_j}}} \right\}, \right. \\ &\quad \left. \left\{ \sqrt{\frac{\prod_{j=1}^n \left(1 + (\eta_j)^2 \right)^{\omega_j} - \prod_{j=1}^n \left(1 - (\eta_j)^2 \right)^{\omega_j}}{\prod_{j=1}^n \left(1 + (\eta_j)^2 \right)^{\omega_j} + \prod_{j=1}^n \left(1 - (\eta_j)^2 \right)^{\omega_j}}} \right\} \right\} \end{aligned} \tag{17}$$

Definition 9 Let $\tilde{d}_j (j = 1, 2, \dots, n)$ be a collection of DHPFNs, then we define the dual hesitant Pythagorean fuzzy Hamacher ordered weighted geometric (DHPFHOWG) operator as follows:

$$\begin{aligned} \text{DHPFHOWG}_w \left(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n \right) &= \bigotimes_{j=1}^n \left(\tilde{d}_{\sigma(j)} \right)^{w_j} \\ &= \cup_{\gamma_{\sigma(j)} \in h_{\sigma(j)}, \eta_{\sigma(j)} \in g_{\sigma(j)}} \left\{ \left\{ \frac{\sqrt{\gamma} \prod_{j=1}^n (\gamma_{\sigma(j)})^{\omega_j}}{\sqrt{\prod_{j=1}^n \left(1 + (\gamma - 1) \left(1 - (\gamma_{\sigma(j)})^2 \right) \right)^{\omega_j} + (\gamma - 1) \prod_{j=1}^n (\gamma_{\sigma(j)})^{2\omega_j}}} \right\}, \right. \\ &\quad \left. \left\{ \sqrt{\frac{\prod_{j=1}^n \left(1 + (\gamma - 1) (\eta_{\sigma(j)})^2 \right)^{w_j} - \prod_{j=1}^n \left(1 - (\eta_{\sigma(j)})^2 \right)^{w_j}}{\prod_{j=1}^n \left(1 + (\gamma - 1) (\eta_{\sigma(j)})^2 \right)^{w_j} + (\gamma - 1) \prod_{j=1}^n \left(1 - (\eta_{\sigma(j)})^2 \right)^{w_j}}} \right\} \right\} \end{aligned} \tag{18}$$

where $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $(1, 2, \dots, n)$, such that $\tilde{d}_{\sigma(j-1)} \geq \tilde{d}_{\sigma(j)}$ for all $j = 2, \dots, n$, and $w = (w_1, w_2, \dots, w_n)^T$ is the aggregation-associated weight vector such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1, \gamma > 0$.

Now, we can discuss some special cases of the DHPFHOWG operator with respect to the parameter γ .

- When $\gamma = 1$, DHPFHOWG operator reduces to the dual hesitant Pythagorean fuzzy ordered weighted geometric (DHPFOWG) operator as follows:

$$\begin{aligned} \text{DHPFOWG}_w(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) &= \bigotimes_{j=1}^n (\tilde{d}_{\sigma(j)})^{w_j} \\ &= \cup_{\gamma_{\sigma(j)} \in h_{\sigma(j)}, \eta_{\sigma(j)} \in g_{\sigma(j)}} \left\{ \left\{ \prod_{j=1}^n (\gamma_{\sigma(j)})^{w_j} \right\}, \left\{ \sqrt{1 - \prod_{j=1}^n (1 - (\eta_{\sigma(j)})^2)^{w_j}} \right\} \right\} \end{aligned} \quad (19)$$

- When $\gamma = 2$, DHPFHOWG operator reduces to the dual hesitant Pythagorean fuzzy Einstein ordered weighted geometric (DHPFEOWG) operator as follows:

$$\begin{aligned} \text{DHPFEOWG}_w(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) &= \bigotimes_{j=1}^n (\tilde{d}_{\sigma(j)})^{w_j} \\ &= \cup_{\gamma_{\sigma(j)} \in h_{\sigma(j)}, \eta_{\sigma(j)} \in g_{\sigma(j)}} \left\{ \left\{ \frac{\sqrt{2} \prod_{j=1}^n (\gamma_{\sigma(j)})^{\omega_j}}{\sqrt{\prod_{j=1}^n (2 - (\gamma_{\sigma(j)})^2)^{\omega_j} + \prod_{j=1}^n (\gamma_{\sigma(j)})^{2\omega_j}}} \right\}, \right. \\ &\quad \left. \left\{ \frac{\prod_{j=1}^n (1 + (\eta_{\sigma(j)})^2)^{w_j} - \prod_{j=1}^n (1 - (\eta_{\sigma(j)})^2)^{w_j}}{\prod_{j=1}^n (1 + (\eta_{\sigma(j)})^2)^{w_j} + \prod_{j=1}^n (1 - (\eta_{\sigma(j)})^2)^{w_j}} \right\} \right\} \end{aligned} \quad (20)$$

From Definitions 8 and 9, we know that the DHPFHWG operator weights the dual hesitant Pythagorean fuzzy argument itself, while the DHPFHOWG operator weights the ordered positions of the dual hesitant Pythagorean fuzzy arguments instead of weighting the arguments themselves. Therefore, weights represent different aspects in both the DHPFHWG and DHPFHOWG operators. However, both the operators consider only one of them. To solve this drawback, in the following we shall propose a dual hesitant Pythagorean fuzzy Hamacher hybrid geometric (DHPFHGG) operator.

Definition 10 A dual hesitant Pythagorean fuzzy Hamacher hybrid geometric (DHPFHHG) operator is defined as follows:

$$\begin{aligned}
 DHPFHHG_{w,\omega}(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) &= \bigotimes_{j=1}^n (\tilde{d}_{\sigma(j)})^{w_j} \\
 &= \cup_{\gamma_{\sigma(j)} \in h_{\sigma(j)}, \dot{\eta}_{\sigma(j)} \in g_{\sigma(j)}} \left\{ \left\{ \frac{\sqrt{\gamma} \prod_{j=1}^n (\dot{\gamma}_{\sigma(j)})^{\omega_j}}{\sqrt{\prod_{j=1}^n (1 + (\gamma - 1) (1 - (\dot{\gamma}_{\sigma(j)})^2))^{\omega_j} + (\gamma - 1) \prod_{j=1}^n (\dot{\gamma}_{\sigma(j)})^{2\omega_j}}} \right\}, \right. \\
 &\quad \left. \left\{ \sqrt{\frac{\prod_{j=1}^n (1 + (\gamma - 1) (\dot{\eta}_{\sigma(j)})^2)^{\omega_j} - \prod_{j=1}^n (1 - (\dot{\eta}_{\sigma(j)})^2)^{\omega_j}}{\prod_{j=1}^n (1 + (\gamma - 1) (\dot{\eta}_{\sigma(j)})^2)^{\omega_j} + (\gamma - 1) \prod_{j=1}^n (1 - (\dot{\eta}_{\sigma(j)})^2)^{\omega_j}}} \right\} \right\} \tag{21}
 \end{aligned}$$

where $w = (w_1, w_2, \dots, w_n)$ is the associated weighting vector, with $w_j \in [0, 1]$, $\sum_{j=1}^n w_j = 1$, and $h_{\sigma(j)}$ is the j -th largest element of the dual hesitant Pythagorean fuzzy arguments $\tilde{d}(\tilde{d} = (\tilde{d}_j)^{n\omega_j}, j = 1, 2, \dots, n)$, $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ is the weighting vector of dual hesitant Pythagorean fuzzy arguments $\tilde{d}_j (j = 1, 2, \dots, n)$, with $\omega_j \in [0, 1]$, $\sum_{j=1}^n \omega_j = 1$, and n is the balancing coefficient, $\gamma > 0$. Especially, if $w = (1/n, 1/n, \dots, 1/n)^T$, then DHPFHHG is reduced to the dual hesitant Pythagorean fuzzy weighted geometric (DHPFHWG) operator; if $\omega = (1/n, 1/n, \dots, 1/n)$, then DHPFHHG is reduced to the dual hesitant Pythagorean fuzzy ordered weighted geometric (DHPFHOWG) operator.

From Definition 10, we know that:

- When $\gamma = 1$, DHPFHHG operator reduces to the dual hesitant Pythagorean fuzzy hybrid geometric (DHPFHG) operator as follows:

$$\begin{aligned}
 DHPFHG_{w,\omega}(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) &= \bigotimes_{j=1}^n (\tilde{d}_{\sigma(j)})^{w_j} \\
 &= \cup_{\gamma_{\sigma(j)} \in h_{\sigma(j)}, \dot{\eta}_{\sigma(j)} \in g_{\sigma(j)}} \left\{ \left\{ \prod_{j=1}^n (\dot{\gamma}_{\sigma(j)})^{w_j} \right\}, \left\{ \sqrt{1 - \prod_{j=1}^n (1 - (\dot{\eta}_{\sigma(j)})^2)^{w_j}} \right\} \right\} \tag{22}
 \end{aligned}$$

- When $\gamma = 2$, DHPFHG operator reduces to the dual hesitant Pythagorean fuzzy Einstein hybrid geometric (DHPFEHG) operator as follows:

$$\begin{aligned}
 & \text{DHPFEHG}_{w,\omega}(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) = \bigotimes_{j=1}^n (\tilde{d}_{\sigma(j)})^{\omega_j} \\
 & = \cup_{\gamma_{\sigma(j)} \in h_{\sigma(j)}, \eta_{\sigma(j)} \in g_{\sigma(j)}} \left\{ \left\{ \frac{\sqrt{2} \prod_{j=1}^n (\gamma_{\sigma(j)})^{\omega_j}}{\sqrt{\prod_{j=1}^n (2 - (\gamma_{\sigma(j)})^2)^{\omega_j} + \prod_{j=1}^n (\gamma_{\sigma(j)})^{2\omega_j}}} \right\}, \right. \\
 & \left. \left\{ \sqrt{\frac{\prod_{j=1}^n (1 + (\eta_{\sigma(j)})^2)^{\omega_j} - \prod_{j=1}^n (1 - (\eta_{\sigma(j)})^2)^{\omega_j}}{\prod_{j=1}^n (1 + (\eta_{\sigma(j)})^2)^{\omega_j} + \prod_{j=1}^n (1 - (\eta_{\sigma(j)})^2)^{\omega_j}}} \right\} \right\} \quad (23)
 \end{aligned}$$

4. Dual hesitant Pythagorean fuzzy Hamacher power operators

4.1. Dual hesitant Pythagorean fuzzy Hamacher power Hamacher power weighted average (DHPFHPWA) operator

Yager [43] developed a nonlinear weighted average aggregation operator called power average (PA) operator, which can be defined as follows:

$$PA(a_1, a_2, \dots, a_n) = \frac{\sum_{i=1}^n (1 + T(a_i)) a_i}{\sum_{i=1}^n (1 + T(a_i))} \quad (24)$$

where $T(a_i) = \sum_{\substack{j=1 \\ j \neq i}}^n Sup(a_i, a_j)$, and $Sup(a, b)$ is the support for a from b , which satisfies

the following three properties:

- (1) $Sup(a, b) \in [0, 1]$;
- (2) $Sup(a, b) = Sup(b, a)$;
- (2) $Sup(a, b) \geq Sup(x, y)$.

In this section, we shall propose the dual hesitant Pythagorean fuzzy Hamacher power weighted average (DHPFHPWA) operator based on the power average [43] operators and Hamacher operations [36].

Definition 11 Let $\tilde{d}_j (j = 1, 2, \dots, n)$ be a collection of DHPFNs, then we define the dual hesitant Pythagorean fuzzy Hamacher power weighted average (DHPFHPWA) operator as follows:

$$DHPFHPWA(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) = \bigoplus_{j=1}^n \left(\frac{\omega_j (1 + T(\tilde{d}_j)) \tilde{d}_j}{\sum_{j=1}^n \omega_j (1 + T(\tilde{d}_j))} \right) = \cup_{\gamma_j \in h_j, \eta_j \in g_j}$$

$$\left\{ \left(\frac{\prod_{j=1}^n \left(1 + (\gamma - 1)(\gamma_j)^2 \right)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}} - \prod_{j=1}^n \left(1 - (\gamma_j)^2 \right)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}}}{\sqrt{\prod_{j=1}^n \left(1 + (\gamma - 1)(\gamma_j)^2 \right)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}} + (\gamma - 1) \prod_{j=1}^n \left(1 - (\gamma_j)^2 \right)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}}}} \right), \right.$$

$$\left. \left(\frac{\sqrt{\gamma} \prod_{j=1}^n (\eta_j)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}}}{\sqrt{\prod_{j=1}^n \left(1 + (\gamma - 1)(1 - (\eta_j)^2) \right)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}} + (\gamma - 1) \prod_{j=1}^n (\eta_j)^{\frac{2\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}}}} \right) \right\} \quad (25)$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of $\tilde{d}_j (j = 1, 2, \dots, n)$, $\gamma > 0$, and

$$T(\tilde{d}_j) = \sum_{\substack{i=1 \\ i \neq j}}^n \omega_i \text{Sup}(\tilde{d}_j, \tilde{d}_i) \quad (26)$$

and $\text{Sup}(\tilde{d}_j, \tilde{d}_i)$ is the support for \tilde{d}_j from \tilde{d}_i , with the conditions:

- $\text{Sup}(\tilde{d}_i, \tilde{d}_j) \in [0, 1]$;
- $\text{Sup}(\tilde{d}_i, \tilde{d}_j) = \text{Sup}(\tilde{d}_j, \tilde{d}_i)$;
- $\text{Sup}(\tilde{d}_i, \tilde{d}_j) \geq \text{Sup}(\tilde{d}_s, \tilde{d}_t)$, if $\text{dis}(\tilde{d}_i, \tilde{d}_j) \geq \text{dis}(\tilde{d}_s, \tilde{d}_t)$, where dis is a distance measure.

Now, we can discuss some special cases of the DHPFHPWA operator with respect to the parameter γ .

- When $\gamma = 1$, DHPFHPWA operator reduces to the dual hesitant Pythagorean fuzzy power weighted average (DHPFPWA) operator as follows:

$$\begin{aligned}
 & \text{DHPFPWAA}_\omega(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) \\
 &= \bigoplus_{j=1}^n \left(\frac{\omega_j (1 + T(\tilde{d}_j)) \tilde{d}_j}{\sum_{j=1}^n \omega_j (1 + T(\tilde{d}_j))} \right) \\
 &= \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \left\{ \sqrt{1 - \prod_{j=1}^n (1 - (\gamma_j)^2)^{\omega_j(1+T(\tilde{d}_j))} / \sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))} \right\}, \right. \\
 & \quad \left. \left\{ \prod_{j=1}^n (\eta_j)^{\omega_j(1+T(\tilde{d}_j))} / \sum_{j=1}^n \omega_j(1+T(\tilde{d}_j)) \right\} \right\}
 \end{aligned} \tag{27}$$

- When $\gamma = 2$, DHPFHPWA operator reduces to the dual hesitant Pythagorean fuzzy Einstein power weighted average (DHPFEPWA) operator as follows:

$$\begin{aligned}
 & \text{DHPFEPWA}_\omega(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) = \bigoplus_{j=1}^n \left(\frac{\omega_j (1 + T(\tilde{d}_j)) \tilde{d}_j}{\sum_{j=1}^n \omega_j (1 + T(\tilde{d}_j))} \right) = \cup_{\gamma_j \in h_j, \eta_j \in g_j} \\
 & \left\{ \left\{ \frac{\prod_{j=1}^n (1 + (\gamma_j)^2)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}} - \prod_{j=1}^n (1 - (\gamma_j)^2)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}}}{\prod_{j=1}^n (1 + (\gamma_j)^2)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}} + \prod_{j=1}^n (1 - (\gamma_j)^2)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}}}, \right. \\
 & \quad \left. \left\{ \frac{\sqrt{2} \prod_{j=1}^n (\eta_j)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}}}{\sqrt{\prod_{j=1}^n (2 - (\eta_j)^2)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}} + \prod_{j=1}^n (\eta_j)^{\frac{2\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}}} \right\} \right\}
 \end{aligned} \tag{28}$$

4.2. Dual hesitant Pythagorean fuzzy Hamacher power weighted geometric (DHPFHPWG) operator

Xu and Yager [44] developed power geometric (PG) operator on the basis of PA operator [43] and geometric mean [45-46], which can be defined as follows:

$$PG(a_1, a_2, \dots, a_n) = \prod_{i=1}^n (a_i)^{(1+T(a_i))} / \sum_{i=1}^n (1+T(a_i)) \tag{29}$$

where $T(a_i) = \sum_{\substack{j=1 \\ j \neq i}}^n Sup(a_i, a_j)$, and $Sup(a, b)$ is the support for a from b , which satisfies the following three properties:

- (1) $Sup(a, b) \in [0, 1]$;
- (2) $Sup(a, b) = Sup(b, a)$;
- (3) $Sup(a, b) \geq Sup(x, y)$, if $|a - b| < |x - y|$.

In this section, we shall propose the dual hesitant Pythagorean fuzzy Hamacher power weighted geometric (DHPFHPWG) operator based on the power geometric [44] operators and Hamacher operations [36].

Definition 12 Let $\tilde{d}_j (j = 1, 2, \dots, n)$ be a collection of DHPFNs, then we define the dual hesitant Pythagorean fuzzy Hamacher power weighted geometric (DHPFHPWG) operator as follows:

$$DHPFHPWG(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) = \bigoplus_{j=1}^n (\tilde{d}_j) = \cup_{\gamma_j \in h_j, \eta_j \in g_j}$$

$$\left\{ \left(\sqrt{\gamma \prod_{j=1}^n (\gamma_j)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}}} \right) \right. \\ \left. \sqrt{\prod_{j=1}^n \left(1 + (\gamma - 1) \left(1 - (\gamma_j)^2 \right) \right)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}}} + (\gamma - 1) \prod_{j=1}^n (\gamma_j)^{\frac{2\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}}} \right\}, \\ \left\{ \left(\sqrt{\prod_{j=1}^n \left(1 + (\gamma - 1) (\eta_j)^2 \right)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}}} - \prod_{j=1}^n \left(1 - (\eta_j)^2 \right)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}}} \right) \right. \\ \left. \sqrt{\prod_{j=1}^n \left(1 + (\gamma - 1) (\eta_j)^2 \right)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}}} + (\gamma - 1) \prod_{j=1}^n \left(1 - (\eta_j)^2 \right)^{\frac{2\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}}} \right\} \tag{30}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of $\tilde{d}_j (j = 1, 2, \dots, n)$, $\gamma > 0$, and

$$T(\tilde{d}_j) = \sum_{\substack{i=1 \\ i \neq j}}^n \omega_i \text{Sup}(\tilde{d}_j, \tilde{d}_i) \tag{31}$$

and $\text{Sup}(\tilde{d}_j, \tilde{d}_i)$ is the support for \tilde{d}_j from \tilde{d}_i , with the conditions:

- (1) $\text{Sup}(\tilde{d}_i, \tilde{d}_j) \in [0, 1]$;
- (2) $\text{Sup}(\tilde{d}_i, \tilde{d}_j) = \text{Sup}(\tilde{d}_j, \tilde{d}_i)$;
- (3) $\text{Sup}(\tilde{d}_i, \tilde{d}_j) \geq \text{Sup}(\tilde{d}_s, \tilde{d}_t)$, if $\text{dis}(\tilde{d}_i, \tilde{d}_j) \geq \text{dis}(\tilde{d}_s, \tilde{d}_t)$ where dis is a distance measure.

Now, we can discuss some special cases of the DHPFHPWG operator with respect to the parameter γ .

- When $\gamma = 1$, DHPFHPWG operator reduces to the dual hesitant Pythagorean fuzzy power weighted geometric (DHPFPWG) operator as follows:

$$\begin{aligned} \text{DHPFPWG}_\omega(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) &= \bigotimes_{j=1}^n (\tilde{d}_j)^{\omega_j(1+T(\tilde{d}_j))} / \sum_{j=1}^n \omega_j(1+T(\tilde{d}_j)) \\ &= \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \left\{ \prod_{j=1}^n (\gamma_j)^{\omega_j(1+T(\tilde{d}_j))} / \sum_{j=1}^n \omega_j(1+T(\tilde{d}_j)) \right\}, \right. \\ &\quad \left. \left\{ \sqrt{1 - \prod_{j=1}^n (1 - (\eta_j)^2)^{\omega_j(1+T(\tilde{d}_j))} / \sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))} \right\} \right\} \end{aligned} \tag{32}$$

- When $\gamma = 2$, DHPFHPWG operator reduces to the dual hesitant Pythagorean fuzzy Einstein power weighted geometric (DHPFEPWG) operator as follows:

$$\begin{aligned} \text{DHPFEPWG}_\omega(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) &= \bigoplus_{j=1}^n (\tilde{d}_j) = \cup_{\gamma_j \in h_j, \eta_j \in g_j} \\ &\quad \left\{ \left(\frac{\sqrt{2} \prod_{j=1}^n (\gamma_j)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}}}{\sqrt{\prod_{j=1}^n (2 - (\gamma_j)^2)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}} + \prod_{j=1}^n (\gamma_j)^{\frac{2\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}}} \right) \right\}, \end{aligned}$$

$$\left\{ \left[\frac{\prod_{j=1}^n \left(1 + (\eta_j)^2 \right)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}} - \prod_{j=1}^n \left(1 - (\eta_j)^2 \right)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}}}{\prod_{j=1}^n \left(1 + (\eta_j)^2 \right)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}} + \prod_{j=1}^n \left(1 - (\eta_j)^2 \right)^{\frac{\omega_j(1+T(\tilde{d}_j))}{\sum_{j=1}^n \omega_j(1+T(\tilde{d}_j))}}} \right] \right\} \quad (33)$$

5. An approach to multiple attribute decision making with dual hesitant Pythagorean fuzzy information

In this section, we shall utilize the dual hesitant aggregation operators to multiple attribute decision making with dual hesitant Pythagorean fuzzy information. Let $A = \{A_1, A_2, \dots, A_m\}$ be a discrete set of alternatives, and $G = \{G_1, G_2, \dots, G_n\}$ be the state of nature. If the decision makers provide several values for the alternative A_i under the state of nature G_j with anonymity, these values can be considered as a dual hesitant Pythagorean fuzzy element $\tilde{d}_{ij} = (h_{ij}, g_{ij})$. In the case where two decision makers provide the same value, then the value emerges only once in \tilde{d}_{ij} . Suppose that the decision matrix $\tilde{D} = (\tilde{d}_{ij})_{m \times n}$ is the dual hesitant Pythagorean fuzzy decision matrix, where $\tilde{d}_{ij} (i = 1, 2, \dots, m, j = 1, 2, \dots, n)$ are in the form of DHPFNs.

In the following, we apply the DHPFHWA (or DHPFHWG) operator to the MADM problems for potential evaluation of emerging technology commercialization with dual hesitant Pythagorean fuzzy information.

Step 1 We utilize the decision information given in matrix \tilde{D} , and the DHPFHWA operator

$$\begin{aligned} \tilde{d}_i &= \text{DHPFHWA} \left(\tilde{d}_{i1}, \tilde{d}_{i2}, \dots, \tilde{d}_{in} \right) \\ &= \bigoplus_{j=1}^n \left(\omega_j \tilde{d}_{ij} \right) \\ &= \cup_{\gamma_{ij} \in h_{ij}, \eta_{ij} \in g_{ij}} \left\{ \left[\frac{\prod_{j=1}^n \left(1 + (\gamma_{ij})^2 \right)^{\omega_j} - \prod_{j=1}^n \left(1 - (\gamma_{ij})^2 \right)^{\omega_j}}{\prod_{j=1}^n \left(1 + (\gamma_{ij})^2 \right)^{\omega_j} + (\gamma_{ij}) \prod_{j=1}^n \left(1 - (\gamma_{ij})^2 \right)^{\omega_j}} \right], \right. \\ &\quad \left. \left[\frac{\sqrt{\gamma} \prod_{j=1}^n (\eta_{ij})^{\omega_j}}{\sqrt{\prod_{j=1}^n \left(1 + (\gamma_{ij}) \left(1 - (\eta_{ij})^2 \right) \right)^{\omega_j} + (\gamma_{ij}) \prod_{j=1}^n (\eta_{ij})^{2\omega_j}}} \right] \right\}, \quad (34) \end{aligned}$$

Or the dual hesitant Pythagorean fuzzy weighted geometric (DHPFHWG) operator:

$$\begin{aligned}
 \tilde{d}_i &= \text{DHPFHWG} \left(\tilde{d}_{i1}, \tilde{d}_{i2}, \dots, \tilde{d}_{in} \right) \\
 &= \bigotimes_{j=1}^n \left(\tilde{d}_{ij} \right)^{\omega_j} \\
 &= \cup_{\gamma_{ij} \in h_{ij}, \eta_{ij} \in \tilde{h}_{ij}} \left\{ \left\{ \frac{\sqrt{\gamma} \prod_{j=1}^n (\gamma_{ij})^{\omega_j}}{\sqrt{\prod_{j=1}^n \left(1 + (\gamma - 1) \left(1 - (\gamma_{ij})^2 \right) \right)^{\omega_j} + (\gamma - 1) \prod_{j=1}^n (\gamma_{ij})^{2\omega_j}}} \right\}, \right. \\
 &\quad \left. \left\{ \frac{\prod_{j=1}^n \left(1 + (\gamma - 1) (\eta_{ij})^2 \right)^{\omega_j} - \prod_{j=1}^n \left(1 - (\eta_{ij})^2 \right)^{\omega_j}}{\sqrt{\prod_{j=1}^n \left(1 + (\gamma - 1) (\eta_{ij})^2 \right)^{\omega_j} + (\gamma - 1) \prod_{j=1}^n \left(1 - (\eta_{ij})^2 \right)^{\omega_j}}} \right\} \right\}
 \end{aligned} \quad (35)$$

to derive the overall preference values \tilde{d}_i ($i = 1, 2, \dots, m$) of the alternative A_i .

Step 2 Calculate the scores $S(\tilde{d}_i)$ ($i = 1, 2, \dots, m$) of the overall dual hesitant Pythagorean fuzzy preference values \tilde{d}_i ($i = 1, 2, \dots, m$). If there is no difference between two scores $S(\tilde{d}_i)$ and $S(\tilde{d}_j)$, then we need to calculate the accuracy degrees $S(\tilde{p}_i)$ and $S(\tilde{p}_j)$ of the collective overall preference values \tilde{d}_i and \tilde{d}_j , respectively, and then rank the alternatives A_i and A_j in accordance with the accuracy degrees $p(\tilde{d}_i)$ and $p(\tilde{d}_j)$.

Step 3 Rank all the alternatives A_i ($i = 1, 2, \dots, m$) and select the best one(s) in accordance with the scores $S(\tilde{d}_i)$ ($i = 1, 2, \dots, m$).

Step 3 End.

6. Numerical example

Thus, in this section we shall present a numerical example for supplier selection in supply chain management with dual hesitant Pythagorean fuzzy information in order to illustrate the method proposed in this paper. Let us suppose there is a problem to deal with the supplier selection in supply chain management which is classical multiple attribute decision making problems. There are five prospect suppliers A_i ($i = 1, 2, 3, 4, 5$) for four attributes G_j ($j = 1, 2, 3, 4$). The four attributes include product quality (G_1), service (G_2), delivery (G_3) and price (G_4), respectively. In order to avoid influence each

other, the decision makers are required to evaluate the five suppliers $A_i (i = 1, 2, 3, 4, 5)$ under the above four attributes in anonymity and the decision matrix $\tilde{D} = (\tilde{d}_{ij})_{5 \times 4}$ is presented in Tab. 1, where $\tilde{d}_{ij} (i = 1, 2, 3, 4, 5, j = 1, 2, 3, 4)$ are in the form of DHPFNs.

Table 1: Dual hesitant Pythagorean fuzzy decision matrix

	G_1	G_2	G_3	G_4
A1	{{0.3,0.4},{0.6}}	{{0.4,0.5},{0.3,0.4}}	{{0.2,0.3},{0.7}}	{{0.4,0.5},{0.5}}
A2	{{0.6},{0.4}}	{{0.2,0.4,0.5},{0.4}}	{{0.2},{0.6,0.7,0.8}}	{{0.5},{0.4,0.5}}
A3	{{0.5,0.7},{0.2}}	{{0.2},{0.7,0.8}}	{{0.2,0.3,0.4},{0.6}}	{{0.5,0.6,0.7},{0.3}}
A4	{{0.7},{0.3}}	{{0.6,0.7,0.8},{0.2}}	{{0.1,0.2},{0.3}}	{{0.1},{0.6,0.7,0.8}}
A5	{{0.6,0.7},{0.2}}	{{0.2,0.3,0.4},{0.5}}	{{0.4,0.5},{0.2}}	{{0.2,0.3,0.4},{0.5}}

The information about the attribute weights is known as follows: $\omega = (0.20, 0.15, 0.35, 0.30)$. In the following, we utilize the approach developed for supplier selection in supply chain management with dual hesitant Pythagorean fuzzy information.

Step 1 We utilize the decision information given in matrix \tilde{D} , and the DHPFHWA operator to obtain the overall preference values \tilde{d}_i of the supplier in supply chain management $A_i (i = 1, 2, 3, 4, 5)$. Take alternative A_i for an example (here, we take $\gamma = 3$), we have

$$\begin{aligned} \tilde{d}_1 &= \text{DHPFHWA}_\omega(\tilde{d}_{11}, \tilde{d}_{12}, \tilde{d}_{13}, \tilde{d}_{14}) \\ &= \bigoplus_{j=1}^4 (\omega_j \tilde{d}_{1j}) \\ &= \cup_{\gamma_{1j} \in h_{1j}, \eta_{1j} \in h_{1j}} \left\{ \left\{ \sqrt{\frac{\prod_{j=1}^4 (1 + (\gamma - 1)(\gamma_{1j})^2)^{\omega_j} - \prod_{j=1}^4 (1 - (\gamma_{1j})^2)^{\omega_j}}{\prod_{j=1}^4 (1 + (\gamma - 1)(\gamma_{1j})^2)^{\omega_j} + (\gamma - 1) \prod_{j=1}^4 (1 - (\gamma_{1j})^2)^{\omega_j}}} \right\}, \right. \\ &\quad \left. \left\{ \frac{\sqrt{\gamma} \prod_{j=1}^4 (\eta_{1j})^{\omega_j}}{\sqrt{\prod_{j=1}^4 (1 + (\gamma - 1)(1 - (\eta_{1j})^2))^{\omega_j} + (\gamma - 1) \prod_{j=1}^4 (\eta_{1j})^{2\omega_j}}} \right\} \right\} \\ &= \{ \{ \{ 0.3, 0.4 \}, \{ 0.5 \} \}, \{ \{ 0.4, 0.5 \}, \{ 0.3, 0.4 \} \}, \{ \{ 0.2, 0.3 \}, \{ 0.5 \} \}, \{ \{ 0.4, 0.5 \}, \{ 0.5 \} \} \} \\ &= \{ \{ 0.3005, 0.3146, 0.3281, 0.3328, 0.3333, 0.3412, 0.3457, 0.3461, 0.3582, 0.3586, \\ &\quad 0.3630, 0.3703, 0.3707, 0.3749, 0.3866, 0.3979 \}, \{ 0.3097, 0.3416 \} \} \end{aligned}$$

Step 2 Calculate the scores $s(\tilde{d}_i)$ ($i = 1, 2, 3, 4, 5$) of the overall dual hesitant Pythagorean fuzzy preference values \tilde{d}_i ($i = 1, 2, 3, 4, 5$):

$$s(\tilde{d}_1) = 0.3828, s(\tilde{d}_2) = 0.4552, s(\tilde{d}_3) = 0.5008$$

$$s(\tilde{d}_4) = 0.4774, s(\tilde{d}_5) = 0.6171$$

Step 3 Rank all the suppliers A_i ($i = 1, 2, 3, 4, 5$) in accordance with the scores $s(\tilde{d}_i)$ ($i = 1, 2, 3, 4, 5$) of the overall dual hesitant Pythagorean fuzzy numbers: $A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$, and thus the most desirable supplier is A_5 .

Based on the DHPFHWG operator, then, in order to select the most desirable supplier, we can develop an approach to multiple attribute decision making problems with dual hesitant Pythagorean fuzzy information, which can be described as following:

Step 1' Aggregate all dual hesitant Pythagorean fuzzy value \tilde{h}_{ij} ($j = 1, 2, 3, 4$) by using the dual hesitant Pythagorean fuzzy weighted geometric (DHPFHWG) operator to derive the overall dual hesitant Pythagorean fuzzy values \tilde{d}_i ($i = 1, 2, \dots, 5$) of the supplier A_i . Take supplier A_1 for an example (here, we take $\gamma = 3$), we have

$$\begin{aligned} \tilde{d}_1 &= \text{DHPFHWG}_\omega(\tilde{d}_{11}, \tilde{d}_{12}, \tilde{d}_{13}, \tilde{d}_{14}) \\ &= \bigotimes_{j=1}^4 (\tilde{d}_{1j})^{\omega_j} \\ &= \cup_{\gamma_{1j} \in h_{1j}, \eta_{1j} \in h_{1j}} \left\{ \left\{ \frac{\sqrt{\gamma} \prod_{j=1}^4 (\gamma_{1j})^{\omega_j}}{\sqrt{\prod_{j=1}^4 \left(1 + (\gamma - 1) \left(1 - (\gamma_{1j})^2\right)\right)^{\omega_j} + (\gamma - 1) \prod_{j=1}^4 (\gamma_{1j})^{2\omega_j}}} \right\}, \right. \\ &\quad \left. \left\{ \frac{\prod_{j=1}^4 \left(1 + (\gamma - 1) (\eta_{1j})^2\right)^{\omega_j} - \prod_{j=1}^4 \left(1 - (\eta_{1j})^2\right)^{\omega_j}}{\sqrt{\prod_{j=1}^4 \left(1 + (\gamma - 1) (\eta_{1j})^2\right)^{\omega_j} + (\gamma - 1) \prod_{j=1}^4 \left(1 - (\eta_{1j})^2\right)^{\omega_j}}} \right\} \right\} \\ &= \{ \{ \{0.3, 0.4\}, \{0.5\} \}, \{ \{0.4, 0.5\}, \{0.3, 0.4\} \}, \{ \{0.2, 0.3\}, \{0.5\} \}, \{ \{0.4, 0.5\}, \{0.5\} \} \} \\ &= \{ \{0.2794, 0.2863, 0.2934, 0.3006, 0.3053, 0.3128, 0.3204, 0.3275, 0.3282, 0.3354, \\ &\quad 0.3435, 0.3518, 0.3572, 0.3657, 0.3743, 0.3832\}, \{0.5913, 0.6031\} \} \end{aligned}$$

Step 2' Calculate the scores $s(\tilde{d}_i)$ ($i = 1, 2, 3, 4, 5$) of the overall dual hesitant Pythagorean fuzzy values \tilde{d}_i ($i = 1, 2, 3, 4, 5$) of the supplier A_i :

$$s(\tilde{d}_1) = 0.4862, s(\tilde{d}_2) = 0.4084, s(\tilde{d}_3) = 0.4076$$

$$s(\tilde{d}_4) = 0.4941, s(\tilde{d}_5) = 0.5502$$

Step 3' Rank all the suppliers in supply chain management A_i ($i = 1, 2, 3, 4, 5$) in accordance with the scores $s(\tilde{d}_i)$ ($i = 1, 2, 3, 4, 5$) of the overall dual hesitant Pythagorean fuzzy values \tilde{d}_i ($i = 1, 2, \dots, 5$) by using definition 5: $A_5 \succ A_4 \succ A_1 \succ A_2 \succ A_3$ and thus the most desirable supplier in supply chain management is A_5 .

From the above analysis, it is easily seen that although the overall rating values of the alternatives are slightly different by using two operators respectively. However, the most desirable supplier in supply chain management is A_5 .

7. Conclusion

In this paper, we investigate the multiple attribute decision making (MADM) problem based on the Hamacher aggregation operators with dual Pythagorean hesitant fuzzy information. Then, motivated by the ideal of Hamacher operation, we have developed some Hamacher aggregation operators for aggregating dual hesitant Pythagorean fuzzy information: dual hesitant Pythagorean fuzzy Hamacher weighted average (DHPFHWA) operator, dual hesitant Pythagorean fuzzy Hamacher weighted geometric (DHPFHWG) operator, dual hesitant Pythagorean fuzzy Hamacher ordered weighted average (DHPFHWA) operator, dual hesitant Pythagorean fuzzy Hamacher ordered weighted geometric (DHPFHOWG) operator, dual hesitant Pythagorean fuzzy Hamacher hybrid average (DHPFHHA) operator and dual hesitant Pythagorean fuzzy Hamacher hybrid geometric (DHPFHGG) operator. The prominent characteristic of these proposed operators are studied. Then, we have utilized these operators to develop some approaches to solve the dual hesitant Pythagorean fuzzy multiple attribute decision making problems. Finally, a practical example for supplier selection in supply chain management is given to verify the developed approach and to demonstrate its practicality and effectiveness. In the future, we shall continue working in the extension and application of the developed operators to other domains and uncertain environments [47-66].

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