www.czasopisma.pan.pl PAN www.journals.pan.pl

10.2478/v10170-010-0016-5

Archives of Control Sciences
Volume 20(LVI), 2010
No. 3*, pages 253–266*

Observer state feedback control of discrete-time systems with state equality constraints

ANNA FILASOVÁ and DUŠAN KROKAVEC

Design conditions for existence of observer-based memory-less feedback control for stabilization of discrete-time systems with equality constraints given on the state variables are presented in the paper. These conditions are associated with eigenstructure assignment based on the singular value decomposition principle. The validity of the proposed method is demonstrated by a numerical example with the equality constraint tying together all state variables.

Key words: equality constraints, eigenstructure assignment, state feedback

1. Introduction

In the last years many significant results have spurred interest in the problem of determining the control laws for the systems with constraints. For the typical case where a system state reflects a certain physical entities this class of constraints rises because of physical limits and these ones usually keep the system state in a region of the technological conditions. Some authors deal with the problem of this kind that designing a control law such that states be driven to origin asymptotically while the coordinates of the command input are subject to unsymmetrical or symmetrical constraints [1], and [4], [3], [14], [16], respectively, others prefer respect of constraints by constructing a stabilizing memory-less controller with inequality defined on the control law gain matrix [2], [3]. Special attention is also focused on the principle of Kalman filtering with equality and inequality state constraints [5], [8], where it is possible to reduce the system model, and use the reduced state equation for such systems, and for given linear state equality constraints.

However, this problem can be formulated using technique dealing with the state constraints directly, where the equations of both, the unconstraint system and the stabilized

The Authors are with Technical University of Košice, Faculty of Electrical Engineering and Informatics, Department of Cybernetics and Artificial Intelligence, Košice, Slovak Republic, fax: +421 55 602 2564, e-mails: anna.filasova@tuke.sk, dusan.krokavec@tuke.sk

The work presented in this paper was supported by VEGA, Grant Agency of Ministry of Education and Academy of Science of Slovak Republic under Grant No. 1/0328/08. This support is very gratefully acknowledged.

Received 4.08.2010.

constraint relations are combined to a coupled system of equations which can be interpreted as a descriptor system [6]. Because a system with state constraints generally does not satisfy the conditions under which the results of descriptor systems can be applicable this approach is limited in a realization.

In principle, it is possible and ever easy to apply a direct design method, namely to design a controller that stabilizes the systems and simultaneously forces the closed-loop systems to satisfy the constraint such that a special form of the constrained problems can be so formulated while the system state variables satisfy the equality constraints [13]. This technique for discrete-time has been introduced in [7] and was extensively used in the reconfigurable control design [10], [11].

The task considered in the paper is to design observer state feedback control of discrete-time linear systems that forces selected state variables of a linear system to satisfy prescribed equality constraint relation and guarantees control system asymptotical stability. Paper extends the problem of control design with state equality constraints [12], and reformulates eigenstructure assignment design principle to obtain desired pole placement of the closed loop system.

2. Problem formulation

Through the paper the task is concerned with design of the state feedback (3) which controls a discrete-time linear dynamic system given by the set of state equations

$$\boldsymbol{q}(i+1) = \boldsymbol{F}\boldsymbol{q}(i) + \boldsymbol{G}\boldsymbol{u}(i) \tag{1}$$

$$\boldsymbol{z}(i) = \boldsymbol{C}\boldsymbol{q}(i) \tag{2}$$

where $\boldsymbol{q}(i) \in I\!R^n$, $\boldsymbol{u}(i) \in I\!R^r$, and $\boldsymbol{y}(i) \in I\!R^m$ are vectors of the state, input and output variables, respectively, and matrices $\boldsymbol{F} \in I\!R^{n\times n}$, $\boldsymbol{G} \in I\!R^{n\times r}$, and $\boldsymbol{C} \in I\!R^{m\times n}$ are real matrices. Problem of the interest is to design an asymptotically stable closed-loop system using a linear memoryless state feedback controller of the form

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}_e(i) \tag{3}$$

where $\boldsymbol{q}_e(i) \in I\!R^n$ is the estimated system state vector, $\boldsymbol{K} \in I\!R^{r\times n}$ is the feedback controller gain matrix, and design constraint in the next equality form

$$\boldsymbol{q}(i) \in \mathcal{N}_{\boldsymbol{D}} = \{\boldsymbol{q} : \boldsymbol{D}\boldsymbol{q} = \boldsymbol{0};\ \boldsymbol{q} \in I\!R^n\} \tag{4}$$

is considered, with $\boldsymbol{D} \in I\!R^{k\times n}$, $\mathrm{rank}\boldsymbol{D} = k \leqslant r$.

3. Preliminaries

Proposition 1 (Matrix pseudoinverse) *Let $\boldsymbol{\Theta}$ is a matrix variable and $\boldsymbol{A}$, $\boldsymbol{B}$ are known non-square matrices of appropriate dimensions such the equality*

$$\boldsymbol{B}\boldsymbol{\Theta} = \boldsymbol{A} \tag{5}$$

can be set. Then all solution to $\boldsymbol{\Theta}$ means

$$\boldsymbol{\Theta} = \boldsymbol{B}^{\ominus 1}\boldsymbol{A} + (\boldsymbol{I} - \boldsymbol{B}^{\ominus 1}\boldsymbol{B})\boldsymbol{\Theta}^{\circ} \tag{6}$$

where

$$\boldsymbol{B}^{\ominus 1} = \boldsymbol{B}^{T}(\boldsymbol{B}\boldsymbol{B}^{T})^{-1} \tag{7}$$

is Moore-Penrose pseudoinverse of $\boldsymbol{B}$ and $\boldsymbol{\Theta}^{\circ}$ is an arbitrary matrix of appropriate dimension.

Proof. Supposing that the product $\boldsymbol{B}\boldsymbol{B}^{T}$ is a regular matrix, then pre-multiplying left-hand side of (5) by the identity matrix gives

$$\boldsymbol{B}\boldsymbol{\Theta} = \boldsymbol{B}\boldsymbol{B}^{T}(\boldsymbol{B}\boldsymbol{B}^{T})^{-1}\boldsymbol{A} \tag{8}$$

and with (7) it yields

$$\boldsymbol{\Theta} = \boldsymbol{B}^{T}(\boldsymbol{B}\boldsymbol{B}^{T})^{-1}\boldsymbol{A} = \boldsymbol{B}^{\ominus 1}\boldsymbol{A} \tag{9}$$

Let $\boldsymbol{\Theta}^{\circ}$ is another matrix of appropriate dimension such that substituting in (5) it can be written

$$\boldsymbol{B}\boldsymbol{\Theta}^{\circ} = \boldsymbol{B}\boldsymbol{B}^{\ominus 1}\boldsymbol{A} = \boldsymbol{B}\boldsymbol{B}^{\ominus 1}\boldsymbol{B}\boldsymbol{\Theta}^{\circ} \tag{10}$$

Thus,

$$\boldsymbol{B}(\boldsymbol{I} - \boldsymbol{B}^{\ominus 1}\boldsymbol{B})\boldsymbol{\Theta}^{\circ} \doteq \boldsymbol{0} \tag{11}$$

$$(\boldsymbol{I} - \boldsymbol{B}^{\ominus 1}\boldsymbol{B})\boldsymbol{\Theta}^{\circ} \doteq \boldsymbol{0} \tag{12}$$

respectively. Therefore, for an arbitrary $\boldsymbol{\Theta}^{\circ}$ (9), (12) implies (6). Note, pseudoinverse is generalized for a singular matrix $\boldsymbol{B}\boldsymbol{B}^{T}$. This concludes the proof. □

Proposition 2 *Let $\boldsymbol{H} \in \mathbb{R}^{n \times n}$ is a real square matrix with non-repeated eigenvalues, satisfying the equality constraint*

$$\boldsymbol{d}^{T}\boldsymbol{H} = 0 \tag{13}$$

Then one from its eigenvalues is zero, and (normalized) $\boldsymbol{d}^{T}$ is the left raw eigenvector of $\boldsymbol{H}$ associated with the zero eigenvalue.

Proof. If $\boldsymbol{H} \in \mathbb{R}^{n \times n}$ is a real square matrix having non-repeated eigenvalues the eigenvalue decomposition of $\boldsymbol{H}$ takes the form

$$\boldsymbol{H} = \boldsymbol{N}\boldsymbol{Z}\boldsymbol{M}^{T} \tag{14}$$

$$\boldsymbol{N} = \begin{bmatrix} \boldsymbol{n}_1 & \cdots & \boldsymbol{n}_n \end{bmatrix}, \ \boldsymbol{M} = \begin{bmatrix} \boldsymbol{m}_1 & \cdots & \boldsymbol{m}_n \end{bmatrix}, \ \boldsymbol{Z} = \operatorname{diag}\begin{bmatrix} z_1 & \cdots & z_n \end{bmatrix} \tag{15}$$

$$\boldsymbol{M}^T\boldsymbol{N} = \boldsymbol{I} \tag{16}$$

where $\boldsymbol{n}_l$, is right eigenvector, and $\boldsymbol{m}_l^T$ is left eigenvector associated with the eigenvalue z_l of $\boldsymbol{H}$, $l = 1,2,\ldots n$. Then (13) can be rewritten as

$$\boldsymbol{d}^T \begin{bmatrix} \boldsymbol{n}_1 & \cdots & \boldsymbol{n}_h & \cdots \boldsymbol{n}_n \end{bmatrix} \operatorname{diag}\begin{bmatrix} z_1 & \cdots & z_h & \cdots & z_n \end{bmatrix} \boldsymbol{M}^T = \boldsymbol{0} \tag{17}$$

If $\boldsymbol{d}^T = \boldsymbol{m}_h^T$ then orthogonal property (16) implies

$$\begin{bmatrix} \boldsymbol{0}_1 & \cdots & \boldsymbol{1}_h & \cdots \boldsymbol{0}_n \end{bmatrix} \operatorname{diag}\begin{bmatrix} z_1 & \cdots & z_h & \cdots & z_n \end{bmatrix} \boldsymbol{M}^T = \boldsymbol{0} \tag{18}$$

and it is evident that (18) can be satisfied only if $z_h = 0$. This concludes the proof. □

Proposition 3 (Eigenstructure assignment) *Considering system (1), (2), and a prescribed closed-loop system matrix eigenvalue spectrum* $\rho(\boldsymbol{F}_c) = \{z_l : |z_l| < 1, \ l = 1,2,\ldots,n\}$, $\rho(\boldsymbol{F}_c) \cap \rho(\boldsymbol{F}) = 0$, *then the gain matrix* $\boldsymbol{K}$ *of the control law*

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}(i) \tag{19}$$

can be designed using column vectors of null spaces of the matrices

$$\mathcal{N}_{z_l} = \mathcal{N}\begin{bmatrix} z_l\boldsymbol{I} - \boldsymbol{F} & \boldsymbol{G} \end{bmatrix}, \quad l = 1,2,\ldots,n \tag{20}$$

Proof. Introducing the closed-loop system matrix $\boldsymbol{F}_c$ as

$$\boldsymbol{F}_c = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{K} \tag{21}$$

then, if $\boldsymbol{n}_h$ is the right eigenvector corresponding to the eigenvalue $z_h \in \rho(\boldsymbol{F}_c)$, it yields

$$(\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K})\boldsymbol{n}_h = z_h\boldsymbol{n}_h \tag{22}$$

and (22) can now be rewritten in the following singular form

$$\begin{bmatrix} z_h\boldsymbol{I} - \boldsymbol{F} & \boldsymbol{G} \end{bmatrix} \begin{bmatrix} \boldsymbol{n}_h \\ \boldsymbol{K}\boldsymbol{n}_h \end{bmatrix} = \boldsymbol{L}_h \begin{bmatrix} \boldsymbol{n}_h \\ \boldsymbol{K}\boldsymbol{n}_h \end{bmatrix} = \boldsymbol{0} \tag{23}$$

Subsequently, the singular value decomposition (SVD) of $\boldsymbol{L}_h$ gives

$$\begin{gathered} \begin{bmatrix} \boldsymbol{u}_{h1}^T \\ \vdots \\ \boldsymbol{u}_{hn}^T \end{bmatrix} \boldsymbol{L}_h \begin{bmatrix} \boldsymbol{v}_{h1} & \cdots & \boldsymbol{v}_{hn} & \boldsymbol{v}_{h,n+1} & \cdots & \boldsymbol{v}_{h,n+r} \end{bmatrix} = \\ = \begin{bmatrix} \sigma_{h1} & & & & & \\ & \ddots & & \boldsymbol{0}_{n+1} & \cdots & \boldsymbol{0}_{n+r} \\ & & \sigma_{hn} & & & \end{bmatrix} \end{gathered} \tag{24}$$

where $\{\boldsymbol{u}_{hl}^T,\ l=1,2,\ldots,n\}$, $\{\boldsymbol{v}_{hl},\ l=1,2,\ldots,n+r\}$ are sets of the left and the right singular vectors of $\boldsymbol{L}_h$, respectively and $\{\sigma_{hl},\ l=1,2,\ldots,n\}$ is a set of the singular values of $\boldsymbol{L}_h$.

It is evident that all column vectors $\{\boldsymbol{v}_{hl},\ l=n+1,n+2,\ldots,n+r\}$ satisfy (23) (the set of these orthonormal column vectors is a non-trivial solution of (23)), and results the null space of $\boldsymbol{L}_h$, i.e.

$$\begin{bmatrix} \boldsymbol{n}_h \\ \boldsymbol{K}\boldsymbol{n}_h \end{bmatrix} \in \mathcal{N}\begin{bmatrix} z_h\boldsymbol{I}_n-\boldsymbol{F} & \boldsymbol{G} \end{bmatrix} \tag{25}$$

which implies (20). This concludes the proof. □

Remark 1 If all prescribed closed-loop eigenvalues are real and non-repeated, $\rho(\boldsymbol{F}_c)\cap\rho(\boldsymbol{F})=0$, and null spaces associated with the closed-loop eigenvalue set are noted as

$$\boldsymbol{W}_h=\begin{bmatrix} \boldsymbol{v}_{h,n+1} & \cdots & \boldsymbol{v}_{h,n+r} \end{bmatrix}=\begin{bmatrix} \boldsymbol{w}_{h1} & \cdots & \boldsymbol{w}_h^{\circ} & \cdots & \boldsymbol{w}_{hr} \end{bmatrix},\ h=1,2,\ldots,n \tag{26}$$

where $\boldsymbol{w}_h^{\circ}=\boldsymbol{w}_{hk}$ is an arbitrary selected column vector from this null space associated with the prescribed eigenvalue z_h, then selecting one column vector from each null space it is possible to construct a matrix $\boldsymbol{W}$, and to partition it using (25) as follows

$$\boldsymbol{W}=\begin{bmatrix} \boldsymbol{w}_1^{\circ} & \boldsymbol{w}_2^{\circ} & \cdots & \boldsymbol{w}_n^{\circ} \end{bmatrix}=\begin{bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & \cdots & \boldsymbol{p}_n \\ \boldsymbol{K}\boldsymbol{p}_1 & \boldsymbol{K}\boldsymbol{p}_2 & \cdots & \boldsymbol{K}\boldsymbol{p}_n \end{bmatrix}=\begin{bmatrix} \boldsymbol{P} \\ \boldsymbol{K}\boldsymbol{P} \end{bmatrix}=\begin{bmatrix} \boldsymbol{P} \\ \boldsymbol{Q} \end{bmatrix} \tag{27}$$

Thus, the gain matrix $\boldsymbol{K}$ can be computed as

$$\boldsymbol{K}=\boldsymbol{Q}\boldsymbol{P}^{-1} \tag{28}$$

Since the spectrum $\rho(\boldsymbol{F}_c)$ is closed under complex conjugation, e.g. if the first two desired eigenvalues are complex conjugate, then $\boldsymbol{W}$ is constructed as

$$\boldsymbol{W}=\begin{bmatrix} \mathrm{Re}\,\boldsymbol{w}_1^{\circ} & \mathrm{Im}\,\boldsymbol{w}_1^{\circ} & \cdots & \boldsymbol{w}_n^{\circ} \end{bmatrix} \tag{29}$$

following the same principle of partitioning. Generalization of (27) using repeated eigenvalues of the order up to r can be find e.g. in [9].

4. Constrained observer state control design

4.1. Constrained control

Theorem 1 *Given the system* (1), (2), *and the constraint* (4) *then the constrained control exists if the gain matrix of the control law* (19) *takes the structure*

$$\boldsymbol{K}=\boldsymbol{S}+\boldsymbol{V}\boldsymbol{K}^{\circ} \tag{30}$$

where

$$\boldsymbol{S} = (\boldsymbol{DG})^{\ominus 1}\boldsymbol{DF} \tag{31}$$

$$\boldsymbol{V} = \boldsymbol{I} - (\boldsymbol{DG})^{\ominus 1}\boldsymbol{DG} \tag{32}$$

and $\boldsymbol{K}^\circ \in \mathbb{R}^{r\times n}$ *is such design parameter matrix that* $\boldsymbol{F}_c = \boldsymbol{F} - \boldsymbol{GK}$ *is a stable matrix.*

Proof. (e.g. compare [7], [10]) Using control law (19) the equilibrium control equation (1) takes the form

$$\boldsymbol{q}(i+1) = (\boldsymbol{F} - \boldsymbol{GK})\boldsymbol{q}(i) \tag{33}$$

where the state-variable vectors have to satisfy equalities

$$\boldsymbol{Dq}(i+1) = \boldsymbol{D}(\boldsymbol{F} - \boldsymbol{GK})\boldsymbol{q}(i) = \boldsymbol{0} \tag{34}$$

for $i = 1, 2, \ldots$. Supposing that the matrix $\boldsymbol{K}$ is chosen in such a way that it is satisfied

$$\boldsymbol{D}(\boldsymbol{F} - \boldsymbol{GK}) = \boldsymbol{0} \tag{35}$$

$$\boldsymbol{DF} = \boldsymbol{DGK} \tag{36}$$

respectively, as well as that the closed-loop system matrix $\boldsymbol{F}_c$ is stable (all its eigenvalues lie in the unit circle in the complex plane $\mathcal{Z}$), then the system state stays within the constrain subspace, i.e. $\boldsymbol{q}(i), \boldsymbol{Fq}(i) \in \mathcal{N}_{\boldsymbol{D}}$.

Solving (36) with respect to $\boldsymbol{K}$ then (6) implies all solutions of $\boldsymbol{K}$ as follows

$$\boldsymbol{K} = (\boldsymbol{DG})^{\ominus 1}\boldsymbol{DF} + (\boldsymbol{I} - (\boldsymbol{DG})^{\ominus 1}\boldsymbol{DG})\boldsymbol{K}^\circ \tag{37}$$

where $\boldsymbol{K}^\circ$ is an arbitrary matrix of appropriate dimension. Thus, using (31), (32) the equation (37) conditioned by a stable $\boldsymbol{F}_c$ implies (30)-(32). This concludes the proof. □

Note, $\boldsymbol{V}$ is the projection matrix (the orthogonal projector onto the null space $\mathcal{N}_{\boldsymbol{DG}}$ of $\boldsymbol{DG}$), $\mathcal{N}_{\boldsymbol{D}}$ is the constrain subspace, and the states be constrained in this subspace (the null space of $\boldsymbol{D}$).

Remark 2 Seeking a control policy of the form

$$\boldsymbol{u}(i) = -\boldsymbol{Kq}(i) + \boldsymbol{N}_w\boldsymbol{w}(i) \tag{38}$$

where $\boldsymbol{w}(i) \in \mathbb{R}^{m\times r}$ then (34) implies

$$\boldsymbol{Dq}(i+1) = \boldsymbol{D}(\boldsymbol{F} - \boldsymbol{GK})\boldsymbol{q}(i) + \boldsymbol{DGN}_w\boldsymbol{w}(i) = \boldsymbol{DGN}_w\boldsymbol{w}(i) \tag{39}$$

and it is evident that the system steady state is not zero, but proportional to steady state of $\boldsymbol{w}$.

4.2. Observer state feedback

Theorem 2 *Constrained observer state feedback control of the system* (1), (2) *with the control law* (3), *and with the state estimator*

$$\boldsymbol{q}_e(i+1) = \boldsymbol{F}\boldsymbol{q}_e(i) + \boldsymbol{G}\boldsymbol{u}(i) + \boldsymbol{J}(\boldsymbol{y}(i) - \boldsymbol{y}_e(i)) \tag{40}$$

$$\boldsymbol{y}_e(i) = \boldsymbol{C}\boldsymbol{q}_e(i) \tag{41}$$

corresponding to the same system is satisfied the constraint (4) in the steady-state. Here $\boldsymbol{q}_e(i) \in \mathbb{R}^n$ is a state vector variable estimate, and $\boldsymbol{J} \in \mathbb{R}^{n\times m}$ is the estimator gain matrix designed in such a way that $\boldsymbol{F} - \boldsymbol{J}\boldsymbol{C}$ is a stable matrix.

Proof. Assembling the system state equation (1), (2) and the estimator state equation (40), (41) gives

$$\begin{bmatrix} \boldsymbol{q}(i+1) \\ \boldsymbol{q}_e(i+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{F} & -\boldsymbol{G}\boldsymbol{K} \\ \boldsymbol{J}\boldsymbol{C} & \boldsymbol{F}-\boldsymbol{J}\boldsymbol{C}-\boldsymbol{G}\boldsymbol{K} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(i) \\ \boldsymbol{q}_e(i) \end{bmatrix} \tag{42}$$

It is evident, using the state transformation

$$\begin{bmatrix} \boldsymbol{q}(i) \\ \boldsymbol{e}(i) \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} \boldsymbol{q}(i) \\ \boldsymbol{q}_e(i) \end{bmatrix}, \qquad \boldsymbol{T} = \boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix} \tag{43}$$

that

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{F} & -\boldsymbol{G}\boldsymbol{K} \\ \boldsymbol{J}\boldsymbol{C} & \boldsymbol{F}-\boldsymbol{J}\boldsymbol{C}-\boldsymbol{G}\boldsymbol{K} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}-\boldsymbol{G}\boldsymbol{K} & \boldsymbol{G}\boldsymbol{K} \\ \boldsymbol{0} & \boldsymbol{F}-\boldsymbol{J}\boldsymbol{C} \end{bmatrix} \tag{44}$$

where $\boldsymbol{e}(i) = \boldsymbol{q}(i) - \boldsymbol{q}_e(i)$ is the error between the actual state and the estimated state at time instant i. Thus, these systems are governed by

$$\begin{bmatrix} \boldsymbol{q}(i+1) \\ \boldsymbol{e}(i+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}-\boldsymbol{G}\boldsymbol{K} & \boldsymbol{G}\boldsymbol{K} \\ \boldsymbol{0} & \boldsymbol{F}-\boldsymbol{J}\boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(i) \\ \boldsymbol{e}(i) \end{bmatrix} \tag{45}$$

which implies the separation principle. Defining the congruence transform matrix $\boldsymbol{T}_c$ as follows

$$\boldsymbol{T}_c = \begin{bmatrix} \boldsymbol{D} & \\ & \boldsymbol{I} \end{bmatrix} \tag{46}$$

then multiplying left-hand side of (45) by $\boldsymbol{T}_c$ it is obtained

$$\begin{bmatrix} \boldsymbol{D}\boldsymbol{q}(i+1) \\ \boldsymbol{e}(i+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{D}(\boldsymbol{F}-\boldsymbol{G}\boldsymbol{K}) & \boldsymbol{D}\boldsymbol{G}\boldsymbol{K} \\ \boldsymbol{0} & \boldsymbol{F}-\boldsymbol{J}\boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(i) \\ \boldsymbol{e}(i) \end{bmatrix} \tag{47}$$

Substituting from (34) states

$$\begin{bmatrix} \mathbf{0} \\ \boldsymbol{e}(i+1) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \boldsymbol{DGK} \\ \mathbf{0} & \boldsymbol{F}-\boldsymbol{JC} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(i) \\ \boldsymbol{e}(i) \end{bmatrix} \tag{48}$$

and it is evident that with stable $\boldsymbol{F}_e = \boldsymbol{F}-\boldsymbol{JC}$ in steady-state regime, i.e. when $\boldsymbol{e}(i+1) = \boldsymbol{e}(i) = \mathbf{0}$, the constrained observer state feedback control satisfies (4). This concludes the proof. □

4.3. Constrained control parameter design

Substituting from (30) the closed loop system matrix (21) takes the form

$$\boldsymbol{F}_c = \boldsymbol{F} - \boldsymbol{GK} = \boldsymbol{F} - \boldsymbol{GS} - \boldsymbol{GVK}^\circ = \boldsymbol{F}^\circ - \boldsymbol{GVK}^\circ \tag{49}$$

where

$$\boldsymbol{F}^\circ = \boldsymbol{F} - \boldsymbol{GS} \tag{50}$$

Now, analogously to what was used in (23) the equation (49) can now be rewritten in the next singular vector form

$$\begin{bmatrix} z_h\boldsymbol{I}-\boldsymbol{F}^\circ & \boldsymbol{GV} \end{bmatrix} \begin{bmatrix} \boldsymbol{n}_h \\ \boldsymbol{K}^\circ\boldsymbol{n}_h \end{bmatrix} = \boldsymbol{L}_h^\circ \begin{bmatrix} \boldsymbol{n}_h \\ \boldsymbol{K}^\circ\boldsymbol{n}_h \end{bmatrix} = \mathbf{0} \tag{51}$$

To satisfy the constraint (4) with $\boldsymbol{D} \in \mathbb{R}^{k\times n}$, rank$\boldsymbol{D} = k \leqslant r$ it is evident that the problem is solved if the set of desired eigenvalues takes form

$$\rho(\boldsymbol{F}_c) = \{0, \cdots, 0, z_{k+1}, \cdots, z_n : |z_h| < 1\} \tag{52}$$

where only subset $\rho(\boldsymbol{F}_c^\circ) = \{z_{k+1}, \cdots, z_n : |z_h| < 1\}$, $\rho(\boldsymbol{F}_c^\circ) \cap \rho(\boldsymbol{F}) = 0$, $\rho(\boldsymbol{F}_c) \cap \rho(\boldsymbol{F}) = 0$ can be arbitrary defined.

Thus, the set of matrices $\boldsymbol{L}_h^\circ$,

$$\boldsymbol{L}_h^\circ = \begin{cases} \begin{bmatrix} -\boldsymbol{F}^\circ & \boldsymbol{GV} \end{bmatrix}, & h = 1 \\ \begin{bmatrix} z_h\boldsymbol{I}-\boldsymbol{F}^\circ & \boldsymbol{GV} \end{bmatrix}, & h = k+1, k+2, \ldots n \end{cases} \tag{53}$$

can be used by the same way as in the Remark 1. to compute $\boldsymbol{K}^\circ$ using (27), (28), while taking off k column vectors from the null space of $\boldsymbol{W}_1$ resulting from the SVD of $\boldsymbol{L}_1^\circ$ to construct $\boldsymbol{W}$.

Applying the same method to compute the observer parameter $\boldsymbol{J}$ the duality principle can be used, i.e. the transposition of the estimator system matrix $\boldsymbol{F}_e = \boldsymbol{F}-\boldsymbol{JC}$ is used to formulate the singular problem with respect to $\boldsymbol{J}^T$. Thus, (23) takes now the form

$$\begin{bmatrix} z_h\boldsymbol{I}-\boldsymbol{F}^T & \boldsymbol{C}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{n}_h^\bullet \\ \boldsymbol{J}^T\boldsymbol{n}_h^\bullet \end{bmatrix} = \boldsymbol{L}_h^\bullet \begin{bmatrix} \boldsymbol{n}_h^\bullet \\ \boldsymbol{J}^T\boldsymbol{n}_h^\bullet \end{bmatrix} = \mathbf{0} \tag{54}$$

and with arbitrary chosen elements of the eigenvalue set satisfying $\rho(\boldsymbol{F}_e) = \{z_l^\bullet : |z_l^\bullet| < 1,\ l = 1,2,\ldots,n\}$, $\rho(\boldsymbol{F}_e) \cap \rho(\boldsymbol{F}) = 0$ matrix $\boldsymbol{J}^T$ can be computed by direct application of such strategy as given in the Remark 1.

5. Illustrative example

To demonstrate properties of the proposed approach, the system with two-inputs and two-outputs is used in the example. The parameters of this system are the next

$$\boldsymbol{F} = \begin{bmatrix} 0.9993 & 0.0987 & 0.0042 \\ -0.0212 & 0.9612 & 0.0775 \\ -0.3875 & -0.7187 & 0.5737 \end{bmatrix}, \ \boldsymbol{G} = \begin{bmatrix} 0.0051 & 0.0050 \\ 0.1029 & 0.0987 \\ 0.0387 & -0.0388 \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 0 \end{bmatrix}$$

respectively, for sampling period $Ts = 0.1$ s. The state constraint was specified as

$$\frac{2\,q_1(t) + q_2(t)}{q_3(t)} = 1$$

which implies

$$\boldsymbol{D}_d = \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}, \qquad \boldsymbol{D} = \frac{1}{\|\boldsymbol{D}_d\|}\boldsymbol{D}_d = \begin{bmatrix} 0.8165 & 0.4082 & -0.4082 \end{bmatrix}$$

and subsequently it yields

$$(\boldsymbol{D}\boldsymbol{G})^{\ominus 1} = \begin{bmatrix} 6.6776 \\ 13.2385 \end{bmatrix}$$

$$\boldsymbol{S} = \begin{bmatrix} 6.4470 & 5.1177 & -1.3298 \\ 12.7813 & 10.1460 & -2.6364 \end{bmatrix}, \ \boldsymbol{V} = \begin{bmatrix} 0.7972 & -0.4021 \\ -0.4021 & 0.2028 \end{bmatrix}$$

Specifying the desired eigenvalues spectrum of the closed-loop system matrix to be $\rho(\boldsymbol{F}_c) = \{0, 0.2, 0.5\}$ then by constructing the sequence of $\boldsymbol{L}_h^\circ$, $h = 1,2,3$ taking the structure (53), and computing the associated null-spaces as was introduced in (24), (26) it was obtained there

$$\boldsymbol{W}_1 = \begin{bmatrix} -0.0012 & -0.0013 \\ 0.0215 & 0.0271 \\ 0.1094 & -0.0179 \\ 0.9938 & 0.0014 \\ -0.0000 & 0.9995 \end{bmatrix}, \ W_2 = \begin{bmatrix} 0.0141 & -0.0071 \\ -0.1874 & 0.0945 \\ -0.1592 & 0.0803 \\ 0.9691 & 0.0156 \\ 0.0156 & 0.9921 \end{bmatrix},$$

$$W_3 = \begin{bmatrix} 0.0142 & -0.0072 \\ -0.0948 & 0.0478 \\ -0.0664 & 0.0335 \\ 0.9932 & 0.0034 \\ 0.0034 & 0.9983 \end{bmatrix}$$

From all possible combinations there were selected these vectors to construct the matrix $\boldsymbol{W}$ and its partitions

$$\boldsymbol{W} = \begin{bmatrix} -0.0012 & 0.0141 & 0.0142 \\ 0.0215 & -0.1874 & -0.0948 \\ 0.1094 & -0.1592 & -0.0664 \\ 0.9938 & 0.9691 & 0.9932 \\ -0.0000 & 0.0156 & 0.0034 \end{bmatrix}, \quad \boldsymbol{P} = \begin{bmatrix} -0.0012 & 0.0141 & 0.0142 \\ 0.0215 & -0.1874 & -0.0948 \\ 0.1094 & -0.1592 & -0.0664 \end{bmatrix}, \quad \boldsymbol{Q} = \begin{bmatrix} 0.9938 & 0.9691 & 0.9932 \\ -0.0000 & 0.0156 & 0.0034 \end{bmatrix},$$

and using (28), (30) the next feedback matrices that assigns the desired closed-loop spectrum were recovered

$$\boldsymbol{K}^{\circ} = \begin{bmatrix} 47.5342 & -11.6857 & 11.8880 \\ -0.6756 & -0.1534 & 0.0226 \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} 44.6118 & -4.1362 & 8.1379 \\ -6.4693 & 14.8138 & -7.4119 \end{bmatrix}$$

Note that this selection of $\boldsymbol{W}$ results in the minimal Frobenius norm of $\boldsymbol{K}^{\circ}$.

Analogously, choosing $\rho(\boldsymbol{F}_e) = \rho(\boldsymbol{F}_c)$ (not necessary but possible), constructing the sequence of $\boldsymbol{L}_h^{\bullet}$, $h = 1,2,3$ taking the structure (54), and computing the associated null-spaces as was introduced in (24), (26) the result was

$$\boldsymbol{W}_1^{\bullet} = \begin{bmatrix} -0.1350 & 0.5659 \\ -0.0792 & -0.6020 \\ -0.9477 & -0.0445 \\ 0.2752 & 0.0349 \\ -0.0412 & 0.5606 \end{bmatrix},$$

$$\boldsymbol{W}_2^{\bullet} = \begin{bmatrix} 0.2204 & 0.6266 \\ 0.3387 & -0.6123 \\ 0.8966 & 0.0775 \\ -0.1811 & 0.0079 \\ 0.0026 & 0.4758 \end{bmatrix}$$

$$\boldsymbol{W}_3^{\bullet} = \begin{bmatrix} 0.5118 & 0.6763 \\ 0.5772 & -0.6649 \\ 0.6331 & 0.0390 \\ -0.0468 & 0.0229 \\ 0.0448 & 0.3138 \end{bmatrix}$$

To obtain matrix $\boldsymbol{J}$ with minimal norm so $\boldsymbol{W}^{\bullet}$ was selected in such a way that

$$\boldsymbol{W}^{\bullet} = \begin{bmatrix} -0.1350 & 0.2204 & 0.6763 \\ -0.0792 & 0.3387 & -0.6649 \\ -0.9477 & 0.8966 & 0.0390 \\ 0.2752 & -0.1811 & 0.0229 \\ -0.0412 & 0.0026 & 0.3138 \end{bmatrix}, \quad \boldsymbol{J} = \begin{bmatrix} 0.2596 & 0.2429 \\ 0.2094 & -0.2232 \\ -0.3449 & 0.0276 \end{bmatrix}$$

It can be verifying that the estimator system matrix eigenvalue spectrum is $\rho(\boldsymbol{F}_e)$. The response simulations are given for the control law

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}_e(i) + \boldsymbol{N}_w\boldsymbol{w}(i)$$

where $\boldsymbol{N}_w$ was computed using the static decoupling principle [9], [15], i.e.

$$\boldsymbol{N}_w = [\boldsymbol{C}(\boldsymbol{I}-\boldsymbol{F}+\boldsymbol{G}\boldsymbol{K}\boldsymbol{Z}^{-1}\boldsymbol{J}\boldsymbol{C})^{-1}(\boldsymbol{I}-\boldsymbol{G}\boldsymbol{K}\boldsymbol{Z}^{-1})\boldsymbol{G}]^{-1}, \quad \boldsymbol{Z} = \boldsymbol{I}-\boldsymbol{F}+\boldsymbol{G}\boldsymbol{K}+\boldsymbol{J}\boldsymbol{C}$$

$$\boldsymbol{N}_w = \begin{bmatrix} -6.6785 & 56.4342 \\ 6.7436 & -18.2619 \end{bmatrix}, \quad \boldsymbol{q}(0) = \begin{bmatrix} -0.5 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{q}_e(0) = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}$$

In the presented Figure 1, Figure 2 the example is shown of the closed-loop system response for $\boldsymbol{w}^T(i) = \boldsymbol{0}$, $\boldsymbol{w}^T(i) = [-0.1 \ \ -0.1]$, with $\boldsymbol{D}\boldsymbol{G}\boldsymbol{N}_w\boldsymbol{w}(i) = -0.0818$.

It is evident that the constraint (4) is satisfied at steady state of the system (common variable $q_d(i) = \boldsymbol{D}\boldsymbol{q}(i)$).

6. Concluding remarks

The paper describes a technique for observer state feedback control of discrete-time systems with equality constraints given on state variables. The proposed method poses the problem as a stabilization problem with a static output feedback controller, while the design principle exploits certain degrees of freedom in basic eigenstructure assignment to obtain implementation conditions of the constraint control concept, and its limitations. The validity of the proposed method is verified by a numerical example to demonstrate the role of an equality constraint tying together the state variables.

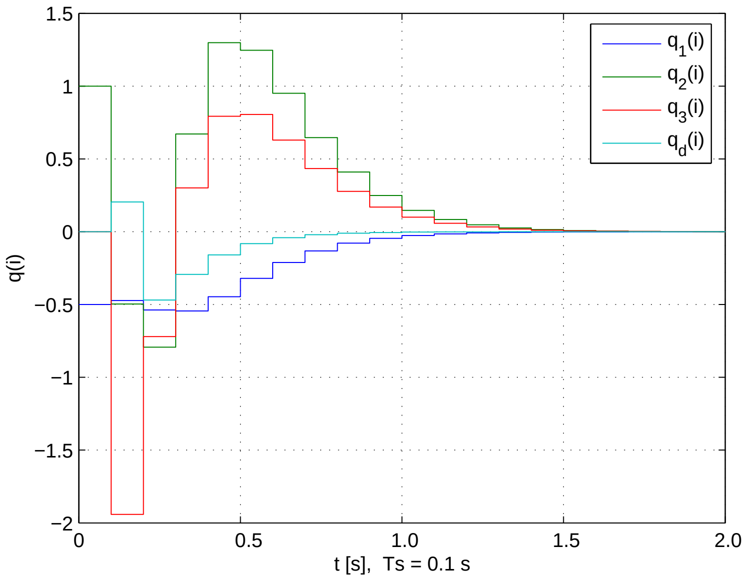

Figure 1. State response of the closed-loop system for $\boldsymbol{w}^T(i) = \mathbf{0}$.

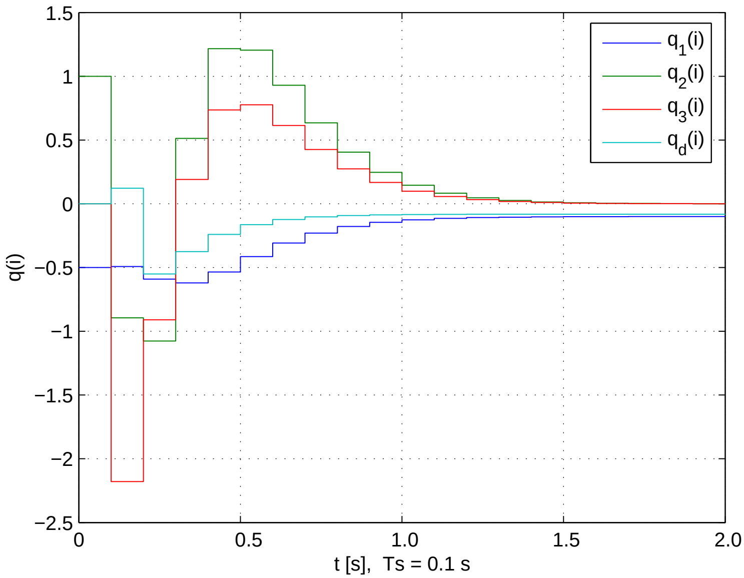

Figure 2. State response of the closed-loop system for $\boldsymbol{w}^T(i) = [-0.1 \ \ -0.1]$.

References

[1] A. BENZAOUIA and C. GURGAT: Regulator problem for linear discrete-time systems with nonsymmetrical constrained control. *Int. J. of Control*, **48**(6), (1988), 2441-2451.

[2] E.B. CASTELAN and J.C. HENNET: Eigenstructure assignment for state constrained linear continuous time systems. *Automatica*, **28**(3), (1992), 605-611.

[3] C.E.T. DÓREA and B.E.A. MILANI: Design of L-Q regulators for state constrained continuous-time systems. *IEEE Trans. on Automatic Control*, **40**(3), (1995), 544-548.

[4] A. FILASOVÁ and D. KROKAVEC: State estimate based control design using the unified algebraic approach. *Archives of Control Sciences*, **20**(1), (2010), 5-18.

[5] N. GUPTA: Kalman filtering in the presence of state space equality constraints. In *Proc. of the 26th Chinese Control Conf.*, Zhangjiajie, China, (2007), 107-113.

[6] H. HAHN: Linear systems controlled by stabilized constraint relations. In *Proc. of the 31st IEEE Conf. on Decision and Control*, Tucson, Arizona, (1992), 840-848.

[7] S. KO and R.R. BITMEAD: Optimal control for linear systems with state equality constraints. *Automatica*, **43**(9), (2007), 1573–1582.

[8] S. KO and R.R. BITMEAD: State estimation for linear systems with state equality constraints. *Automatica*, **43**(9), (2007), 1363 1368.

[9] D. KROKAVEC and A. FILASOVÁ: Discrete–time Systems. Elfa, Košice, 2008, (in Slovak).

[10] D. KROKAVEC and A. FILASOVÁ: Performance of reconfiguration structures based on the constrained control. In *Proc. of the 17th IFAC World Congress*, Seoul, Korea, (2008), 1243-1248.

[11] D. KROKAVEC and A. FILASOVÁ: Control reconfiguration based on the constrained LQ control algorithms. In *Prep. of the 7th IFAC Symp. on Fault Detection, Supervision and Safety of Technical Processes SAFEPROCESS*, Barcelona, Spain, (2009), 686-691.

[12] D. KROKAVEC and A. FILASOVÁ: State constrained control design. In *Proc. of the 11th International Carpathian Control Conf. ICCC 2100.*, Eger, Hungary, (2010), 489-492.

[13] H. OLOOMI and B. SHAFAI: Constrained stabilization problem and transient mismatch phenomenon in singularity perturbed systems. *Int. J. of Control*, **67**(2), (1997), 435-454.

[14] S. TARBOURIECH and E.B. CASTELAN: An eigenstructure assignment approach for constrained linear continuous-time singular systems. *Systems & Control Letters*, **24**(5), (1995), 333-343.

[15] Q.G. WANG: Decoupling Control. Springer-Verlag, Berlin, 2003.

[16] Y. XUE, Y. WEI and G. DUAN: Eigenstructure assignment for linear systems with constrained input via state feedback. A parametric approach. In *Proc. of the 25th Chinese Control Conf.*, Harbin, China, (2006), 108-113.

[17] T.J. YU, C.F. LIN and P.C. MÜLLER: Design of LQ regulator for linear systems with algebraic-equation constraints, In *Proc. of the 35th Conf. on Decision and Control*, Kobe, Japan, (1996), 4146-4151.