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ANALYSIS OF HELICAL COMPRESSION SPRING SUPPORT INFLUENCE ON ITS DEFORMATION

This paper presents a new method of calculation of the change of axial twisting angle of compressed helical spring's end-coils in the case of rotary - free supports. The propriety of derived formulas was experimentally verified. The method is easy in application and gives results much closer to experiment than the presently used method that can be found in literature.

1. Introduction

Helical springs exhibit high sensitivity to supporting conditions [1]. The shape of end coils influences the stiffness of spring. The way of mounting of the spring's end coils influences the susceptibility to buckling [2]. In available literature, one can find formulas allowing calculation of the change of axial twisting angle of statically-compressed helical spring's end-coils [1, 3], but simplifications in these formulas cause that they can not be used for large-deflection cases.

2. Analysis of deformation

Figure 1 shows two types of helical spring supports. Fig.1 a) shows rotary-free support of compress helical spring, Fig.1 b) shows fixed supports preventing ends of the spring from axial twisting. These are two characteristic cases of general support case which acts on spring with axial twisting moment M_o as it is shown in Fig. 2. Compressing force F generates two components of the moment acting on a spring wire:

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$$M_\tau = F\frac{D}{2}\cos\gamma - \text{twisting moment}$$

$$M_N = F\frac{D}{2}\sin\gamma - \text{bending moment}$$

where: γ – the lead angle of spring, D – the nominal diameter of spring.

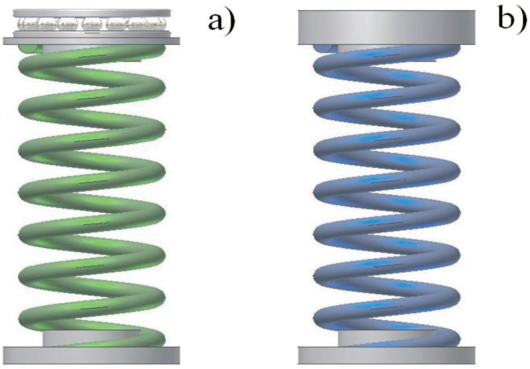

Fig. 1. Two types of helical spring supports: a) rotary-free support, b) fixed support

Fig. 2. Loads on segment of compressed spring

Substituting

$$M = F \cdot \frac{D}{2}$$

We obtain

$$M_\tau = M\cos\gamma$$
$$M_N = M\sin\gamma$$

By adding moments of force F with the axial twisting moment M_o, we obtain the equations of moments:
– wire bending moment

$$M_g = M \sin\gamma - M_o \cos\gamma \tag{1}$$

– wire twisting moment

$$M_s = M \cos\gamma + M_o \sin\gamma \tag{2}$$

Elastic energy of bending equals [4]

$$U_G = \int_0^L \frac{M_g^2}{2EJ} dL \tag{3}$$

Elastic energy of twisting equals

$$U_S = \int_0^L \frac{M_s^2}{2GJ_o} dL \tag{4}$$

Integration is executed over the length of wire. The length of wire equals:

$$L = \frac{\pi D_0}{\cos\gamma_0} z_c \tag{5}$$

where: γ_o – the initial lead angle of spring;
D_o – the nominal diameter of not loaded spring;
z_c – the number of working coils of spring.
The total elastic energy resulting from wire bending and twisting equals

$$U_C = U_G + U_S \tag{6}$$

After transformations and integration, the formula of total elastic energy takes the form

$$U_C = \frac{L}{2EJ}\left(\left(M^2 + M_o^2\right) + \nu\left(M^2\cos^2\gamma + 2MM_o\sin\gamma\cos\gamma + M_o\sin^2\gamma\right)\right) \tag{7}$$

Accordingly to the *Castigliano rule*, the derivative of potential energy with respect to generalised force equals generalised displacement caused by this force [5]:

$$\frac{\partial U}{\partial F_n} = w_n \tag{8}$$

Thus, in general case of spring support, the change of twisting angle of compressed helical spring's end-coils equals:

$$\vartheta = \frac{\partial U_c}{\partial M_o} = \frac{L}{EJ}(M_o + \nu M \sin\gamma\cos\gamma + \nu M_o \sin^2\gamma) \tag{9}$$

In the case of rotary-free support, the twisting moment Mo equals zero. then

$$\vartheta(M_o = 0) = \frac{L}{EJ}\nu M \sin\gamma\cos\gamma \tag{10}$$

Formulas (9) and (10) can be found in literature in a similar form [1, 3]. They can not be very accurate for high deflections of spring because of the assumption that the lead angle is constant during compression of the spring. Let us transform the formula (10) to form (11), which relates elementary rotation $d\vartheta$ to elementary load dF.

For a quite frequent case of mounting element that allows for mutual rotation of spring's end-coils

$$d\vartheta = (\frac{L}{EJ}\nu R \sin\gamma\cos\gamma)dF \tag{11}$$

The height of spring H equals $L\sin\gamma$. For a spring without load, the height H_o equals $L\sin$
$gamma_o$. When loaded, the spring will deflect, with deflection value f , and its height will equal $H = H_o - f$. The stiffness of spring equals $c = \frac{F}{f} \cong idem$.

Thus, transforming foregoing equations, one can write:

$$\sin\gamma = \frac{H_0 - \frac{F}{c}}{L} \tag{12}$$

Hence

$$\cos\gamma = \sqrt{1 - \frac{(H_0 - \frac{F}{c})^2}{L^2}} \tag{13}$$

From geometric relationships between the spring diameter, the lead angle of spring γ and the length of spring wire it results that: $\cos\gamma = \frac{2\pi R \cdot z_c}{L}$,
Thus

$$R = \frac{L}{2\pi z_c}\sqrt{1 - \frac{(H_0 - \frac{F}{c})^2}{L^2}} \tag{14}$$

Substituting (12), (13), (14) into (11) we get:

$$d\vartheta = \frac{L\nu}{EJ} \cdot \frac{L}{2\pi z_c} \sqrt{1 - \frac{(H_0 - \frac{F}{c})^2}{L^2}} \cdot \frac{H_0 - \frac{F}{c}}{L} \cdot \sqrt{1 - \frac{(H_0 - \frac{F}{c})^2}{L^2}} dF$$

Because the inequality $H_0 < L$ always holds, thus, by simplification we get:

$$d\vartheta = \frac{\nu}{2\pi LEJ \cdot z_c}(L^2 - (H_0 - \frac{F}{c})^2) \cdot (H_0 - \frac{F}{c})dF$$

Hence

$$\vartheta = \int_0^{P_M} \frac{\nu}{2\pi LEJ \cdot z_c}(L^2 - (H_0 - \frac{F}{c})^2) \cdot (H_0 - \frac{F}{c})dF$$

After integrating the compressing force F_M from zero to maximum we obtain:

$$\vartheta(M_o = 0) = \frac{F_M \nu}{2\pi LEJ \cdot z_c}\left(H_0 L^2 - H_0^3 + \frac{3H_0^2 F_M}{2c} - H_0\left(\frac{F_M}{c}\right)^2 - \frac{L^2 F_M}{2c} + \frac{1}{4}\left(\frac{F_M}{c}\right)^3\right) \tag{15}$$

Substituting maximum deflection f for F_M/c in (15) yields

$$\vartheta(M_o = 0) = \frac{F_M \nu}{2\pi LEJ \cdot z_c}\left(H_0 L^2 - H_0^3 + \frac{3}{2}H_0^2 f - H_0 f^2 - \frac{1}{2}L^2 f + \frac{1}{4}f^3\right)$$

Formula (15) can be applied to large spring deflections when assumption of constant lead angle of spring γ can not be accepted.

After introducing the following denotation: z_c – initial number of spring coils, z_{cP} – number of spring coils under compressing load F, one can write: $z_{cP} = z_c - \frac{\vartheta}{2\pi}$. The circumference of one spring coil under load equals: $L_{1P} = \frac{L}{z_{cP}}$, hence spring diameter under load equals: $D_P = \frac{L_{1P}}{\pi}\cos\gamma$.

The subscript “P” denotes the parameter under compressing load F of the spring. By substituting the above relationships into the foregoing formula, we finally obtain:

$$D_P = \frac{\sqrt{L^2 - (H_0 - \frac{F}{c})^2}}{z_c \cdot \pi - \frac{\vartheta}{2}} \tag{16}$$

Formula (15) can be used to calculate the angle of mutual rotation of end-coils of loaded spring, whereas formula (16) can be used to calculate the increase of nominal spring diameter under this load.

The propriety of formulas (10) and (15) has been verified experimentally on a spring from car suspension system (Fig. 3). The parameters of the examined spring:

total height l_0 – 395 mm; nominal spring diameter D_{nom} – 119 mm; wire diameter g – 11 mm; number of acting coils n_c – 6,27; spiral lead h – 63 mm

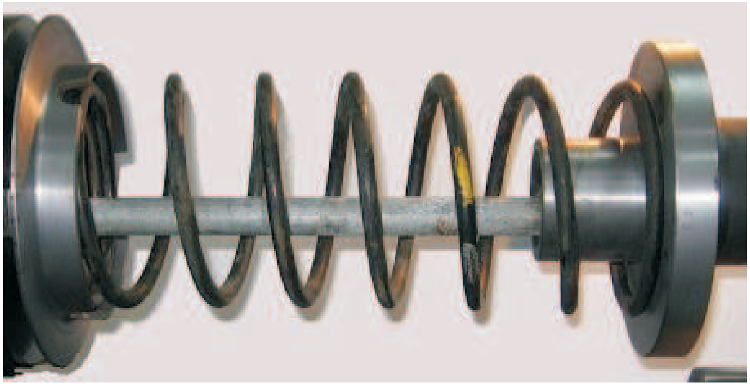

Fig. 3. Examined spring on experiment station

The angle of mutual rotation of end-coils was measured for spring deflection equal to f = 200 mm. The results are shown in Table.1. Material properties of the examined spring were assumed as typical for steel: E = 206000 MPa, ν = 0.3.

Table 1.

Value from:	formula (10)	formula (15)	experiment
ϑ[°]	7.44	5.8	12.7

As one can see in Tab.1, formula (10) gives the result almost two times lower than that obtained in experiment. Moreover, formula (15), which takes into account the change of lead angle gives a result even worse than formula (10),where the change of lead angle is not considered.

A complex investigation was carried out in order to find more precise relations between the results from formula (10) and experiment. In these experiments, we used eighteen springs with different geometrical parameters.

The angle of mutual rotation of end-coils was measured with 0.5[°] accuracy. Bottom support of spring was made of thrust bearing with low moment of friction. Top support of spring was fully constrained.

Fig. 4. Experiment station for complex investigations

The results are shown in Tab. 2.

As one can see in the last column of Tab. 2, the ratio of the experimental value of axial rotation angle to the value of rotation from formula (10) is significantly high.

Formula (10) gives results which are not compatible with experiment because it has been derived using the *Castigliano rule*, which can be applied only for *Clapeyron systems*. A spring loaded with axial force and twisting moment does not fulfil the *Clapeyron system's* conditions when large deflections are considered. Thus, a new formula for mutual rotation of spring end-coils must be derived.

The research was conducted in two stages. In the first stage, the assumption of constant curvature was made. In the second stage, the change of curvature of helix during compression of spring has been considered.

The calculations presented below were done with an assumption that twisting of the spring's wire played the major role in its deflection, whereas bending of wire was only a correcting factor.

The parametric equations of a helix have the form [6]:

$$x = R\cos\varphi; \qquad y = R\sin\varphi; \qquad z = k\varphi \tag{17}$$

where: R – a half of nominal spring diameter, k – increment of coordinate z referring to increment of angle φ from 0 to 2π (Fig. 5). The quantity $2\pi k$ means the spiral lead.

Table 2.

Nr	l_o [mm]	D_{nom} [mm]	g [mm]	h [mm]	n [-]	n_c [-]	f [mm]	ϑ_d [°]	F [N]	ϑ_l [°]	ϑ_d/ϑ_l [-]
1	190	64	10	26.25	8.5	6.5	90	7.5	5280	4.8	1.56
2	87	46.1	6.9	21	6	3.5	48	7.0	3195	4.0	1.75
3	90	39.4	6	14.25	8	5.5	39	4.5	1505	3.0	1.5
4	87	36.7	7.8	16.2	7	4.8	37	4.5	5070	3.7	1.21
5	83	34.5	4	9.8	10	7.8	40	5.0	320	2.8	1.78
6	67	30.8	3.2	10	8.5	6.5	38	4.5	209	3.4	1.32
7	68	24.7	2.3	10.88	8.3	6.3	48	11.0	143	7.2	1.53
8	69	25.5	4	9	9	7	29	4.5	640	3.4	1.32
9	114	32.5	2.5	16	9.5	7.5	84	16.0	127	10.7	1.5
10	315	100	13	36.6	10	8	130	5.0	4640	4.0	1.25
11	320	144	12	103	4.5	3.5	171	11.0	3395	7.0	1.57
12	390	119	11	58	7.5	6	238	14.0	3445	8.9	1.57
13	155	77	15	32	6.1	3.6	65	4.5	20020	2.9	1.55
14	275	90	4	30	10.5	9.5	180	10.0	66	5.37	1.86
15	260	80	4	30	10.5	9.5	180	12.0	95	6.38	1.88
16	275	70	4	30	10.5	9.5	180	13.5	141	8.84	1.53
17	275	60	4	30	10.5	9.5	180	20.0	225	11.96	1.67
18	275	50	4	30	10,5	9.5	180	27.5	388	17.2	1.6
								Average ratio ϑ_d/ϑ_l			1.55

A helix can also be defined by:
– Helix curvature

$$\frac{1}{\rho} = \frac{R}{k^2 + R^2} \tag{18}$$

– Helix torsion

$$\frac{1}{T} = \frac{|k|}{k^2 + R^2} \tag{19}$$

For a certain helix, the curvature and the torsion are constant.
Let's consider a spring with the following parameters:
H_{ocz} – spring height, measured only through acting coils of spring;
n_{0cz} – number of acting coils for spring without load;
h_0, γ_0 – spiral lead and lead angle, respectively, for spring without load;

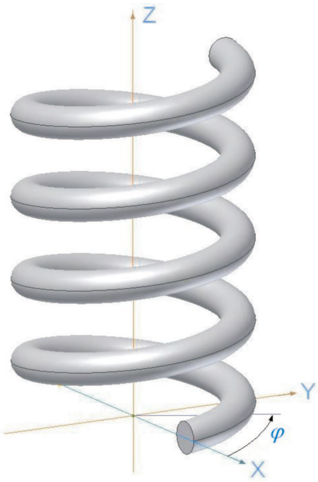

Fig. 5. Analysed spring model

$H_{1cz} = H_{0cz} - f$ – acting height of spring under load.

The spring is placed in the coordinate system, as it is shown in Fig. 5. Thus, the coordinates of bottom spring end are: x = R, y = 0, z = 0.

The developments of helix of a spring under load and of a one without load are presented in Fig. 6.

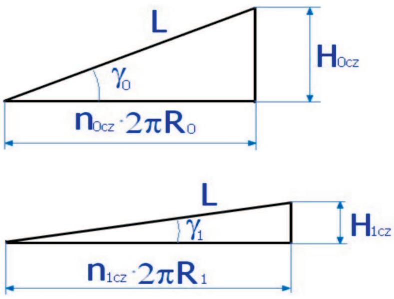

Fig. 6. Developments of helix for two different deflections

The coordinates of top end of the spring without load and that under load equal, respectively,

$$\begin{aligned} x_0 &= R_0 \cos(2\pi \cdot n_{0cz}), \quad y_0 = R_0 \sin(2\pi \cdot n_{0cz}), \quad z_0 = k_0 \cdot (2\pi \cdot n_{0cz}) = H_{0cz} \\ x_1 &= R_1 \cos(2\pi \cdot n_{1cz}), \quad y_1 = R_1 \sin(2\pi \cdot n_{1cz}), \quad z_1 = k_1 \cdot (2\pi \cdot n_{1cz}) = H_{1cz} \end{aligned} \tag{20}$$

Because z_0 is the spring height and $z_0 = H_{0cz}$, thus the coefficient k_0 equals:

$$k_0 = \frac{H_{0cz}}{2\pi \cdot n_{0cz}} \tag{21}$$

On the ground of (18) and (21) one can find the curvature value, which is initially assumed to be constant independently of spring deflection.

$$\frac{1}{\rho} = \frac{R_0}{\left(\frac{H_{0cz}}{2\pi \cdot n_{0cz}}\right)^2 + R_0^2} = idem \tag{22}$$

Knowing the spring's wire curvature value $1/\rho$, one can calculate the value of coefficient k for loaded spring:

$$k_1 = \sqrt{\rho R_1 - R_1^2} \tag{23}$$

On the ground of (21) one can write:

$$k_1 = \frac{H_{1cz}}{2\pi \cdot n_{1cz}} \tag{24}$$

On the ground of Fig. 6. one can write:

$$L^2 = H_{1cz}^2 + (2\pi \cdot n_{1cz} \cdot R_1)^2$$

Hence, the number of acting coils of loaded spring equals:

$$n_{1cz} = \frac{\sqrt{L^2 - H_{1cz}^2}}{2\pi \cdot R_1} \tag{25}$$

By comparing (23) and (24) we obtain

$$\sqrt{\rho R_1 - R_1^2} = \frac{H_{1cz}}{2\pi \cdot n_{1cz}} \tag{26}$$

After substituting (25) into (26) and transforming the formula, we get

$$R_1 = \rho \frac{L^2 - H_{1cz}^2}{L^2} \tag{27}$$

By substituting equation (27) into (25) we obtain, after transformations, the formula for the number of acting coils for loaded spring:

$$n_{1cz} = \frac{\left(H_{0cz}^2 + (2\pi \cdot n_{0cz} \cdot R_0)^2\right) R_0}{2\pi \left(\left(\frac{H_{0cz}}{2\pi \cdot n_{0cz}}\right)^2 + R_0^2\right) \sqrt{H_{0cz}^2 + (2\pi \cdot n_{0cz} \cdot R_0)^2 - H_{1cz}^2}} \tag{28}$$

The angle of mutual rotation of spring's end-coils during its compression is equal to the difference between initial and final torsional angle of the spring:

$$\vartheta = \varphi_0 - \varphi_1 \tag{29}$$

Therefore

$$\vartheta = 2\pi \cdot \left(n_{0cz} - \frac{\left(H_{0cz}^2 + (2\pi \cdot n_{0cz} \cdot R_0)^2\right) R_0}{2\pi \left(\left(\frac{H_{0cz}}{2\pi \cdot n_{0cz}}\right)^2 + R_0^2 \right) \sqrt{H_{0cz}^2 + (2\pi \cdot n_{0cz} \cdot R_0)^2 - H_{1cz}^2}} \right) \tag{30}$$

The foregoing calculations were conducted with an assumption of constant wire curvature. Now, the change of wire curvature during spring compression will be considered. The change of torsion is the result of twisting moment, whilst the change of curvature is the result of bending moment:

$$\frac{1}{T} = f(M_\tau) \qquad \frac{1}{\rho} = f(M_N)$$

The relation between curvature, elastic modulus, moment of inertia and bending moment is as follows [7]:

$$\frac{1}{\rho_z} = \frac{M_N}{EJ} \tag{31}$$

The subscript "z" in formula (31) means that it is a change of curvature, not its entire value. The entire value of curvature is the sum (in the case of tension spring – difference) of initial wire curvature and its change following on spring compression.

$$\frac{1}{\rho} = \frac{1}{\rho_0} + \frac{1}{\rho_z} \tag{32}$$

Transformation of (31) yields

$$\frac{1}{\rho_z} = \frac{FD}{2EJ} \sin\gamma_1 \tag{33}$$

Transforming standard formula for helical spring deflection [8] we get:

$$F = \frac{(H_{0cz} - H_{1cz})Gd^4}{8 \cdot D^3 \cdot n_{cz}} \tag{34}$$

On the ground of Fig. 6 one can write:

$$\sin\gamma_1 = \frac{H_{1cz}}{L} \tag{35}$$

Substituting transformations of formulas (33, 34, 35) into (32) we obtain

$$\frac{1}{\rho_1} = \frac{1}{\rho_0} + \frac{(H_{0cz} - H_{1cz})\,G}{\pi R_0^2 \cdot n_{0cz} \cdot E} \cdot \frac{H_{1cz}}{L} \tag{36}$$

Using the dependence $E/G = 2(1+\nu)$ [7] and equation (22), on the ground of (27) one can write

$$R_{1cz} = \left(\frac{L^2}{L^2 - H_{1cz}^2} \cdot \left(\frac{R_0}{\left(\frac{H_{0cz}}{2\pi \cdot n_{0cz}}\right)^2 + R_0^2} + \frac{(H_{0cz} - H_{1cz})}{2\pi R_0^2 \cdot n_{0cz} \cdot (1+\nu)} \cdot \frac{H_{1cz}}{L} \right) \right)^{-1} \tag{37}$$

Thus, the number of working coils of loaded spring, considering the change of wire curvature, equals

$$n_{1cz} = \frac{L^2}{2\pi\sqrt{L^2 - H_{1cz}^2}} \left(\frac{R_0}{\left(\frac{H_{0cz}}{2\pi \cdot n_{0cz}}\right)^2 + R_0^2} + \frac{(H_{0cz} - H_{1cz})}{2\pi R_0^2 \cdot n_{0cz} \cdot (1+\nu)} \cdot \frac{H_{1cz}}{L} \right) \tag{38}$$

Finally, the dependence for the angle of mutual rotation of the end-coils is given in the form:

$$\vartheta = 2\pi n_{0cz} - \frac{L^2}{\sqrt{L^2 - H_{1cz}^2}} \left(\frac{R_0}{\left(\frac{H_{0cz}}{2\pi \cdot n_{0cz}}\right)^2 + R_0^2} + \frac{(H_{0cz} - H_{1cz})}{2\pi R_0^2 \cdot n_{0cz} \cdot (1+\nu)} \cdot \frac{H_{1cz}}{L} \right) \tag{39}$$

The results of formula (39) were compared with experimental results shown in Tab. 2. The results of the comparison are shown in Tab. 3.

As one can notice, the results are quite coincident, so that formula (39) gives results much more accurate than formula (10) that has been applied to date.

Table 3.

No. Of spring from Tab. 2	The value of twisting angle θ from the experiment in [°]	The value of twisting angle θ from (39) in [°]	The ratio between ϑ from the experiment and from (39)
1	7.5	7.8	0.961
2	7.0	7.1	0.986
3	4.5	4.9	0.918
4	4.5	4.6	0.978
5	5.0	4.4	1.136
6	4.5	5.2	0.865
7	11.0	11.9	0.924
8	4.5	4.9	0.918
9	16.0	17.8	0.9
10	5.0	5.9	0.847
11	11.0	9.6	1.146
12	14.0	15.1	0.927
13	4.5	4.6	0.978
14	10.0	9.3	1.075
15	12.0	11.5	1.043
16	13.5	15.3	0.882
17	20.0	20.7	0.966
18	27.5	29.5	0.932
		Average ratio $\vartheta_d/\vartheta_{(39)}$	0.965

3. Conclusions

In this paper, we have shown that formula (10), still applied in literature, can not be used to calculate the angle of mutual rotation of spring's end-coils in the case of strongly deflected, rotationally-free supported spring. Formula (10) was derived on the basis of assumption that the spring is the *Clapeyron system*. This assumption, however, is not true for the analysed phenomena, because the dependence between the angle of mutual rotation of end-coils and the change of spring's height during compression is not linear. The attempt of reducing simplifying assumptions gave results even more distant from experiment results than those calculated with formula (10).

Formula (39), derived in this paper on the ground of a new approach to the problem, gave much more accurate results, which was verified experimentally. The change of lead angle was taken into consideration. The analysis of formula (10) shows that, when we use it, a negative value of twisting angle can never be obtained. For very high values of lead angle, mutual rotation of spring's end-coils will have a negative value, and such a result can be obtained on the ground of formula (39).

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Analiza wpływu podparcia sprężyny śrubowej naciskowej na jej odkształcenia

S t r e s z c z e n i e

W pracy zaprezentowano nową metodę obliczania kąta skręcenia czół sprężyny śrubowej naciskowej pod obciążeniem dla przypadku obrotowo podatnego podparcia jej końców. Poprawność wyprowadzonych zależności zweryfikowano doświadczalnie. Metoda ta jest prosta w zastosowaniu i daje wyniki znacznie bliższe wynikom eksperymentu niż metoda znana z literatury i dotychczas stosowana.