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Distributed controllability of one-dimensional heat equation in unbounded domains: The Green's function approach

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We derive exact and approximate controllability conditions for the linear one-dimensional heat equation in an infinite and a semi-infinite domains. The control is carried out by means of the time-dependent intensity of a point heat source localized at an internal (finite) point of the domain. By the Green's function approach and the method of heuristic determination of resolving controls, exact controllability analysis is reduced to an infinite system of linear algebraic equations, the regularity of which is sufficient for the existence of exactly resolvable controls. In the case of a semi-infinite domain, as the source approaches the boundary, a lack of L^2-null-controllability occurs, which is observed earlier by Micu and Zuazua. On the other hand, in the case of infinite domain, sufficient conditions for the regularity of the reduced infinite system of equations are derived in terms of control time, initial and terminal temperatures. A sufficient condition on the control time, heat source concentration point and initial and terminal temperatures is derived for the existence of approximately resolving controls. In the particular case of a semi-infinite domain when the heat source approaches the boundary, a sufficient condition on the control time and initial temperature providing approximate controllability with required precision is derived.

Key words: lack of controllability, exact controllability, approximate controllability, null-controllability, Green's function, heuristic method, infinite system of algebraic equations, regularity, fully regularity

1. Introduction

The control systems which are able to accommodate a desired state within a given *finite* amount of time by means of admissible controls are called controllable. Depending on how precisely the desired state is implemented by the

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To the 80th birthday of my father Zhora Khurshudyan, a talented engineer and inventor, is dedicated.

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"best" choice of admissible controls, control systems are classified as exactly or approximately controllable. For a detailed introduction into the concept of controllability and existing methods of analysis, we refer to [1–7].

Mathematically, the controllability analysis can be carried out by evaluating the mismatch between the desired state and the state implemented by a specific choice of the resolving control, i.e., the residue[1]

$$\mathcal{R}_T(\boldsymbol{u}) = \|\boldsymbol{w}(\boldsymbol{u},\boldsymbol{x},T) - \boldsymbol{w}_T(\boldsymbol{x})\|_{\mathbf{W}_T}, \tag{1}$$

where T is the given control time, $\boldsymbol{u}$ is the control function, $\boldsymbol{w}$ is the state function and $\boldsymbol{x}$ is the state variable, $\boldsymbol{w}_T$ is the desired state, $\mathbf{W}_T$ is the space of desired states $\boldsymbol{w}_T$ (an appropriate Hilbert space) endowed with the norm $\|\cdot\|_{\mathbf{W}_T}$.

Thus, if for at least one admissible control

$$\boldsymbol{u} \in \mathcal{U} = \{\boldsymbol{u} \in \mathbf{U},\ \ \mathrm{supp}(\boldsymbol{u}) \subseteq [0, T]\},$$

where $\mathbf{U}$ is the control function space, $\mathrm{supp}(\boldsymbol{u}) = \overline{\{t \in \mathbb{R}^+,\ \boldsymbol{u}(t) \neq 0\}}$ is the support of $\boldsymbol{u}$, the following equality holds for (1):

$$\mathcal{R}_T(\boldsymbol{u}) = 0, \tag{2}$$

then the system is exactly controllable. If this is not the case, but for a *given* $\varepsilon > 0$ and at least one admissible control $\boldsymbol{u} \in \mathcal{U}$,

$$\mathcal{R}_T(\boldsymbol{u}) \leqslant \varepsilon, \tag{3}$$

then the system is called approximately controllable. Accordingly, we define the sets of exactly and approximately resolving controls as follows:

$$\mathcal{U}_{res}^{ex} = \{\boldsymbol{u} \in \mathcal{U},\ (2)\}, \qquad \mathcal{U}_{res}^{ap} = \{\boldsymbol{u} \in \mathcal{U},\ (3)\}. \tag{4}$$

Various factors may lead to the lack of exact or approximate controllability of a particular control system in finite [9–14] or infinite [15] time, i.e., to $\mathcal{U}_{res}^{ex} = \varnothing$ or even $\mathcal{U}_{res}^{ap} = \varnothing$ with a required accuracy. In particular, in [11] it is proved that for arbitrary initial temperature there do not exist L^2-boundary control functions u providing exact null-controllability of the three-dimensional heat equation

$$\frac{\partial\Theta}{\partial t} - \Delta\Theta = 0, \qquad \boldsymbol{x} \in \Omega \subset \mathbb{R}^3, \quad t \in (0, T),$$

subject to the following boundary and initial conditions:

$$\Theta(\boldsymbol{x},t) = u(t)\chi_{\partial\Omega_0}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial\Omega, \quad t \in [0, T],$$

[1] More complicated forms of $\mathcal{R}_T$ that are nonlinear in $\boldsymbol{w}$ can be considered [8].

$$\Theta(\boldsymbol{x}, 0) = \Theta_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \overline{\Omega}.$$

Here Ω is a semi-infinite domain occupied by the heating body, $\partial\Omega$ is its boundary, Θ characterizes the temperature in the body, $\chi_{\partial\Omega_0}$ is the characteristic function of $\partial\Omega_0 \subseteq \partial\Omega$. In the above terminology, this means that

$$\mathcal{R}_T(\boldsymbol{u}) = \|\Theta(\boldsymbol{x}, T)\|_{L^2(\Omega)} \neq 0$$

for all $u \in L^2([0, T])$ and finite T. See also [5, 16–18] and the related references therein.

In [19], a necessary and sufficient condition on initial and terminal states for which there exists an L^1-boundary control providing exact controllability within a finite T of the one-dimensional wave equation with variable coefficients

$$\frac{\partial}{\partial x}\left[N(x)\frac{\partial w}{\partial x}\right] - \rho(x)\frac{\partial^2 w}{\partial t^2} = 0, \qquad x \in \mathbb{R}^+, \quad t \in (0, T),$$

subject to the following boundary and initial conditions:

$$w(0, t) = u(t), \qquad \lim_{x\to\infty} w(x, t) = 0, \qquad t \in [0, T],$$

$$w(x, 0) = w_0(x), \qquad \left.\frac{\partial w}{\partial t}\right|_{t=0} = w_0^1(x), \qquad x \in \mathbb{R}^+.$$

Here $0 \leqslant N \in C^{(1)}(\mathbb{R}^+)$, $0 < \rho \in C(\mathbb{R}^+)$.

In this paper, we consider the distributed exact and approximate controllability for the linear one-diensional heat equation in semi-infinite and infinite domains. The control is carried out by the intensity of a heat source located at an internal point of the domain. Involving the Green's function approach [7, 20, 21] and the heuristic method [22, 23], it becomes possible to reduce the exact controllability in both domains to the analysis of regularity (or fully regularity) of an infinite system of linear algebraic equations. In particular, considering the limiting case when the heat source approaches the boundary of the semi-infinite domain, we derive the result of Micu and Zuazua discussed above [11]. On the other hand, sufficient conditions on the initial and terminal temperatures and the control time are derived for which there exist controls providing exact controllability in an infinite domain. Sufficient conditions for approximate controllability are also derived.

2. Controllability problem

Consider an unbounded, heating rod of sufficiently small thickness providing that at each time instant the temperature of any cross section of the rod has a uniform distribution. Let the rod be heated by a point source with a controllable intensity

$$u \in \mathcal{U} = \left\{u \in L^2[0,T] \cap L^\infty[0,T],\ \ \mathrm{supp}(u) \subseteq [0,T]\right\},$$

concentrated at some internal point $x = x_0 \in \Omega$ of the rod. Here

$$\Omega = \begin{cases} \mathbb{R}, & \text{rod is infinite,} \\ \mathbb{R}^+, & \text{rod is semi-infinite.} \end{cases}$$

In dimensionless variables and quantities (see [18]), the heat transfer in the rod is described by the one-dimensional equation

$$\frac{\partial \Theta}{\partial t} = \alpha \frac{\partial^2 \Theta}{\partial x^2} + u(t)\delta(x - x_0), \qquad x \in \Omega, \quad t \in \mathbb{R}^+, \tag{5}$$

in which Θ is the temperature, $\alpha > 0$ represents physical characteristics of the rod, δ is the Dirac delta function.

The initial temperature distribution in the rod is given by

$$\Theta(x,0) = \Theta_0(x), \quad x \in \Omega. \tag{6}$$

The aim of control is to find all heating regimes $u \in \mathcal{U}$, such that within a given *finite* amount of time T, the state

$$\Theta(x,T) = \Theta_T(x), \quad x \in \Omega \tag{7}$$

is implemented exactly or approximately with a given precision ε.

In the case when $\Omega = \mathbb{R}^+$, we will additionally assume that the $x = 0$ end of the rod is thermo-isolated, so that

$$\Theta(0,t) \equiv 0, \quad t \in \mathbb{R}^+, \tag{8}$$

and that the boundary and initial conditions are consistent, i.e.,

$$\Theta_0(0) = \Theta_T(0) = 0.$$

For the sake of simplicity we assume that $\Theta_0,\ \Theta_T \in L^2(\Omega)$. Then, the problem is to characterize the sets of exactly and approximately resolving controls (4) for the residue

$$\mathcal{R}_T(u) = \|\Theta(x,T) - \Theta_T(x)\|^2_{L^2(\Omega)} = \int\limits_\Omega |\Theta(x,T) - \Theta_T(x)|^2 \,\mathrm{d}x. \tag{9}$$

3. Green's function solution

In order to analyze the controllability of (5), (6), we involve the Green's function approach [7]. Represent the general solution of (5), (6) in terms of the Green's function:

$$\Theta(x,t) = \int_0^t G(x,x_0,t-\tau)u(\tau)\,\mathrm{d}\tau + \int_\Omega G(x,\xi,t)\Theta_0(\xi)\,\mathrm{d}\xi, \tag{10}$$

where [24]

$$G(x,\xi,t) = \begin{cases} \dfrac{1}{\sqrt{4\pi\alpha t}}\exp\left[-\dfrac{(x-\xi)^2}{4\alpha t}\right], & \Omega=\mathbb{R}, \\ \dfrac{1}{\sqrt{4\pi\alpha t}}\left(\exp\left[-\dfrac{(x-\xi)^2}{4\alpha t}\right]-\exp\left[-\dfrac{(x+\xi)^2}{4\alpha t}\right]\right), & \Omega=\mathbb{R}^+. \end{cases}$$

Passing to the limit when $x \to \pm\infty$, it becomes evident that $\Theta \to 0$ uniformly for $t \in \mathbb{R}^+$, $u \in \mathcal{U}$.

In order to make the dependence $\mathcal{R}_T = \mathcal{R}_T(u)$ explicit, we evaluate (10) at $t = T$ and substitute it into (9):

$$\mathcal{R}_T(u) = \int_\Omega \left| \int_0^T G(x,x_0,T-\tau)u(\tau)\,\mathrm{d}\tau + \int_\Omega G(x,\xi,T)\Theta_0(\xi)\,\mathrm{d}\xi - \Theta_T(x) \right|^2 \mathrm{d}x. \tag{11}$$

4. Exact controllability

First, let us examine the exact controllability of (5), (6). Then, by the definition of norm, (11) is equivalent to

$$\int_0^T G(x,x_0,T-\tau)u(\tau)\,\mathrm{d}\tau = M_T(x), \quad x \in \Omega, \tag{12}$$

where

$$M_T(x) = \Theta_T(x) - \int_\Omega G(x,\xi,T)\Theta_0(\xi)\,\mathrm{d}\xi.$$

Let $\{\varphi_n\}_{n=1}^\infty \in L^2(\Omega)$ be a family of functions orthogonal (possibly with some weight) in Ω. Then, (12) is equivalent to the following infinite system of integral constraints on u:

$$\int_0^T G_n(x_0,T-\tau)u(\tau)\,\mathrm{d}\tau = M_{Tn}, \quad n=1,2,\ldots, \tag{13}$$

where

$$G_n(x_0,t)=\int_\Omega G(x,x_0,t)\varphi_n(x)\,\mathrm{d}x, \qquad M_{Tn}=\int_\Omega M_T(x)\varphi_n(x)\,\mathrm{d}x.$$

Thus,

$$\mathcal{U}_{res}^{ex}=\{u\in\mathcal{U},\ (13)\}\,.$$

Let $\{\phi_m\}_{m=1}^{\infty}\in L^2([0,T])$ be a family of functions orthogonal (possibly with some weight) in $[0,T]$. Expanding u into a series of ϕ_m and substituting it into (13), we arrive at the following infinite system of linear algebraic equations with respect to the expansion coefficients:

$$\mathbf{G}(x_0,T)\mathbf{u}=\mathbf{M}_T, \tag{14}$$

where

$$\mathbf{G}(x_0,T)=(G_{nm}(x_0,T))_{m,\ n=1}^{\infty}, \qquad \mathbf{u}=(u_m)_{m=1}^{\infty}, \qquad \mathbf{M}_T=(M_{Tn})_{n=1}^{\infty},$$

$$G_{nm}(x_0,T)=\int_0^T G_n(x_0,T-\tau)\phi_m(\tau)\,\mathrm{d}\tau, \qquad u_m=\int_0^T u(\tau)\phi_m(\tau)\,\mathrm{d}\tau.$$

Solution of (14) is given by the following well-known theorem.

Theorem 1 ([25], p. 27) *If for given $T>0$, x_0 and $\Theta_0,\Theta_T\in L^2(\Omega)$, the infinite system (14) is regular, i.e.,*

$$\sigma_n(x_0,T)=\sum_{m=1}^{\infty}|G_{nm}(x_0,T)|<1, \quad n=1,2,\ldots, \tag{15}$$

and there exists a positive constant C, such that

$$|M_{Tn}|\leqslant C\,[1-\sigma_n(x_0,T)], \quad n=1,2,\ldots, \tag{16}$$

then (14) has a bounded solution

$$|u_m|\leqslant C, \quad m=1,2,\ldots, \tag{17}$$

which can be found by the method of successive approximations.

Moreover, if the solution of the majorant system

$$\sum_{m=1}^{\infty} |G_{nm}(x_0,T)|\, v_m = C\,[1-\sigma_n(x_0,T)], \quad n=1,2,\ldots,$$

satisfies

$$v_m > 0, \quad m=1,2,\ldots, \tag{18}$$

then bounded solution (17) is unique.

Remark 1 *A weaker notion of fully regular systems*[2] *satisfying*

$$\sigma_n(x_0,T) \leqslant 1-c < 1, \quad n=1,2,\ldots, \tag{19}$$

with a positive constant c can be considered. In this case, if (16) is substituted by

$$|M_{Tn}| \leqslant c\cdot C, \quad n=1,2,\ldots, \tag{20}$$

then (17) holds. At this, for fully regular systems, (18) always holds providing that fully regular systems satisfying (20) always have a unique bounded solution.

Remark 2 *The unique solution of regular and fully regular systems are found computing the limit of the unique solution of the truncated $N\times N$ system when $N\to\infty$.*

Thus, we eventually arrive at the following assertion.

Theorem 2 *If for given $T>0$, x_0 and $\Theta_0, \Theta_T \in L^2(\Omega)$, inequalities (15), (16) hold, then $\mathcal{U}^{ex}_{res} = \{u \in \mathcal{U},\ (14)\} \neq \varnothing$, i.e., (5), (6) is exactly controllable. Moreover, if, together with (15), (16), inequality (18) holds as well, or (19), (20) hold, then the exactly resolving control is unique and is given by*

$$u(t) = \sum_{m=1}^{\infty} u_m \phi_m(t), \quad t\in[0,T].$$

4.1. Lack of exact null-controllability in a semi-infinite domain

Let us consider the case when $\Omega=\mathbb{R}^+$. Then, taking as φ_n the Laguerre polynomials L_n which are orthogonal in $\mathbb{R}^+$ with the weight $w(x)=\exp(-x)$, for the exact null-controllability of (5)–(8) we obtain the following system of necessary and sufficient conditions (cf. (13)):

$$\int_0^T G_n(x_0,T-\tau)u(\tau)\,\mathrm{d}\tau = -\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} G(x,\xi,T)\Theta_0(\xi)L_n(x)\exp(-x)\,\mathrm{d}x\,\mathrm{d}\xi, \quad n=1,2,\ldots \tag{21}$$

[2]In the sense that full regularity implies regularity, but the opposite does not always hold.

Making use of Theorem 2, we obtain that if for given $T > 0$, $x_0 > 0$ and $\Theta_0 \in L^2(\mathbb{R}^+)$,

$$\sigma_n(x_0, T) = \sum_{m=1}^{\infty} \left| \int_{\mathbb{R}^+} \int_0^T G(x, x_0, T-\tau) L_n(x) \exp(-x) \phi_m(\tau) \, \mathrm{d}x \, \mathrm{d}\tau \right| < 1$$

for all $n = 1, 2, \ldots$, and

$$\left| \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} G(x, \xi, T) \Theta_0(\xi) L_n(x) \exp(-x) \, \mathrm{d}x \, \mathrm{d}\xi \right| \leqslant C [1 - \sigma_n(x_0, T)]$$

for a positive constant C, then there exists a $u \in \mathcal{U}$ providing the exact null-controllability of (5)–(8) within a required time T.

On the other hand, passing to the limit when $x_0 \to 0$, i.e., the source approaches the boundary, we obtain

$$\lim_{x_0 \to 0} G(x, x_0, t) = 0, \quad x \in \mathbb{R}^+, \quad t \in [0, T],$$

uniformly. In that case, residue (11) takes the following form:

$$\mathcal{R}_T(u) = \int_{\Omega} \left| \int_{\Omega} G(x, \xi, T) \Theta_0(\xi) \, \mathrm{d}\xi \right|^2 \mathrm{d}x,$$

which is independent on u. Therefore, in general, there does not exist an admissible control $u \in \mathcal{U}$ providing exact null-controllability of (5)–(8) in the limiting case $x_0 \to 0$. Recall that for the heat equation, this limiting case has been reported earlier in [11].

4.2. Exact controllability in an infinite domain

Now, let us consider the case when $\Omega = \mathbb{R}$ and $x_0 = 0$ (for the sake of simplicity). This time, as φ_n we take the Hermite polynomials H_n orthogonal in $\mathbb{R}$ with the weight $w(x) = \exp\left(-x^2\right)$. Then,

$$G_n(0, t) = \frac{1}{\sqrt{4\pi\alpha t}} \int_{\mathbb{R}} \exp\left[-\frac{x}{4\alpha t}\right] H_n(x) \exp\left(-x^2\right) \mathrm{d}x =$$

$$= \begin{cases} 0, & n = 2k+1, \\ (-1)^k \dfrac{1}{\sqrt{[1 + 4\pi\alpha t]^{2k+1}}}, & n = 2k. \end{cases}$$

Substituting this into (13), we arrive at the following infinite system of constraints for the control function:

$$(-1)^k \int_0^T \frac{u(\tau)}{\sqrt{[1+4\pi\alpha(T-\tau)]^{2k+1}}}\,\mathrm{d}\tau = M_{T2k}, \quad k = 1,2,\ldots. \tag{22}$$

At this, for the sake of consistency, the initial and required temperatures must satisfy the following system of constraints:

$$M_{T2k+1} = \int_\Omega \left[\Theta_T(x) - \int_\Omega G(x,\xi,T)\Theta_0(\xi)\,\mathrm{d}\xi\right] H_{2k+1}(x)\exp\left(-x^2\right)\mathrm{d}x = 0, \quad k = 1,2,\ldots. \tag{23}$$

Exactly resolving controls can be determined from (22) by the heuristic method [22]. Indeed, let

$$u(t) = \sum_{m=1}^{\infty} u_m \sin\left(\frac{\pi m}{T}t\right), \quad t \in [0,T]. \tag{24}$$

Substituting this expansion into (22), for u_m we derive the following infinite system of linear algebraic equations:

$$\mathbf{G}(T)\mathbf{u} = \mathbf{M}_T, \tag{25}$$

where

$$\mathbf{G}(T) = (G_{km}(T))_{k,m=1}^{\infty},$$
$$G_{km}(T) = (-1)^k \int_0^T \frac{1}{\sqrt{[1+4\pi\alpha(T-\tau)]^{2k+1}}}\sin\left(\frac{\pi m}{T}\tau\right)\mathrm{d}\tau.$$

The regularity or fully regularity of (25) is studied according to Theorem 1. For the exact controllability of the rod it is sufficient that for a given T, the following inequality holds:

$$\sigma_k(T) = \sum_{m=1}^{\infty}\left|\int_0^T \frac{1}{\sqrt{[1+4\pi\alpha\,(T-\tau)]^{2k+1}}}\sin\left(\frac{\pi m}{T}\tau\right)\mathrm{d}\tau\right| < 1.$$

Evaluating the integral and simplifying the absolute value, we derive

$$\sigma_k(T) = \frac{\sqrt{2}}{2^{2(k+1)}\pi\sqrt{\alpha T}} \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \left|\Re\mathfrak{e}\ \mu_{km}(T) - \Im\mathfrak{m}\ \mu_{km}(T)\right|, \tag{26}$$

where

$$\mu_{km}(T) = (-1)^m T \left(\frac{m}{\alpha T} i\right)^k \exp\left[\frac{m}{4\alpha T} i\right] \Delta\Gamma_{km}(T),$$

$$\Delta\Gamma_{km}(T) = \Gamma\left(\frac{1}{2} - k, \left(\pi m + \frac{m}{4\alpha T}\right) i\right) - \Gamma\left(\frac{1}{2} - k, \frac{m}{4\alpha T} i\right)$$

and $\Gamma(\cdot,\cdot)$ is the Euler incomplete Γ-function.

Thus, we have proved the following assertion.

Theorem 3 *If for $T > 0$ and $\Theta_0, \Theta_T \in L^2(\mathbb{R})$, equality (23) holds together with the following inequalities:*

$$|M_{T2k}| \leqslant C[1 - \sigma_k(T)], \quad k = 1, 2, \ldots,$$

for a positive constant C, and

$$\sigma_k(T) < 1, \quad k = 1, 2, \ldots$$

for (26), then the control intensity (24) with (25) ensures the exact controllability of the infinite rod within T.

Remark 3 *Note that equality (23) is satisfied, in particular, when for*

$$F(x) = \int\limits_{\Omega} G(x, \xi, T)\Theta_0(\xi)\, \mathrm{d}\xi,$$

it holds $F, \Theta_T \in \{H_{2k+1}\}_{k=1}^{\infty}$ or $\Theta_T - F \in \{H_{2k+1}\}_{k=1}^{\infty}$.

5. Approximate controllability

Consider now the approximate controllability of the heat equation (5). By virtue of the triangle inequality, (9) is estimated as follows:

$$\sqrt{\mathcal{R}_T(u)} = \|\Theta(x,T) - \Theta_T(x)\|_{L^2(\Omega)}$$
$$\leqslant \left\| \int\limits_0^T G(x, x_0, T - \tau) u(\tau)\, \mathrm{d}\tau \right\|_{L^2(\Omega)} + \|M_T\|_{L^2(\Omega)}.$$

On the other hand,

$$\left\| \int_0^T G(x, x_0, T-\tau) u(\tau)\, \mathrm{d}\tau \right\|_{L^2(\Omega)} \leqslant \int_0^T g(x_0, T-\tau)\, |u(\tau)|\, \mathrm{d}\tau$$

where

$$g(x_0, t) = \|G(x, x_0, t)\|_{L^2(\Omega)}\,.$$

Therefore, we arrive at the following assertion.

Theorem 4 *If for $T > 0$, $\Theta_0, \Theta_T \in L^2(\Omega)$,*

$$\varepsilon_T = \sqrt{\varepsilon} - ||M_T||_{L^2(\Omega)} \geqslant 0,$$

then any admissible control

$$u \in \widetilde{\mathcal{U}}^{ap}_{res} = \left\{ u \in \mathcal{U},\ \int_0^T g(x_0, T-\tau)\, |u(\tau)|\, \mathrm{d}\tau \leqslant \varepsilon_T \right\}$$

is approximately resolving, i.e., $\widetilde{\mathcal{U}}^{ap}_{res} \subseteq \mathcal{U}^{ap}_{res}$.

Remark 4 *Note that ε_T does not depend on x_0.*

Corollary 4 *Under the condition of Theorem 4, any admissible control*

$$u \in \bar{\mathcal{U}}^{ap}_{res} = \left\{ u \in \mathcal{U},\ |u| \leqslant \frac{\varepsilon_T}{\gamma(x_0, T)} \right\}$$

with

$$\gamma(x_0, T) = \int_0^T g(x_0, T-\tau)\, \mathrm{d}\tau,$$

is approximately resolving.

Remark 5 *Obviously, $\bar{\mathcal{U}}^{ap}_{res} \subseteq \widetilde{\mathcal{U}}^{ap}_{res} \subseteq \mathcal{U}^{ap}_{res}$.*

5.1. Approximate null-controllability in a semi-infinite domain

As a particular case, consider the approximate null-controllability of a semi-infinite rod, i.e., let $\Omega = \mathbb{R}^+$, in the limiting case when $x_0 \to 0$. Then, since

$$\lim_{x_0 \to 0} G(x, x_0, t) = 0, \qquad x \in \mathbb{R}^+, \quad t \in \mathbb{R}^+$$

uniformly, the residue will accept the following form:

$$\mathcal{R}_T(u) = \left\| \int_\Omega G(x, \xi, T)\Theta_0(\xi)\, \mathrm{d}\xi \right\|^2_{L^2(\Omega)},$$

which is independent of u. Therefore, as far as for $T > 0$ and $\Theta_0 \in L^2(\Omega)$ the inequality

$$\left\| \int_\Omega G(x, \xi, T)\Theta_0(\xi)\, \mathrm{d}\xi \right\|^2_{L^2(\Omega)} \leqslant \varepsilon$$

holds with a given positive ε, then (5)–(8) is approximately null-controllable. Otherwise, we obtain that it lacks to be approximate null-controllable as well.

6. Conclusion

In this paper, we consider the problem of controllability of the linear one-dimensional heat equation in unbounded domains ($\Omega = \mathbb{R}^+$ and $\Omega = \mathbb{R}$) by means of the intensity (u) of a single heat source localized at an internal point (x_0) of the domain. Given the initial temperature (Θ_0) of the domain, it is required to characterize the set of u which heats the domain until a specified temperature (Θ_T) within a *finite* amount of time (T) exactly or approximately.

By the Green's function approach and the heuristic method for determination of resolving controls, the exact controllability is reduced to the analysis of regularity or fully regularity of an infinite system of linear algebraic equations. The regularity analysis of the infinite system corresponding to a semi-infinite rod allows to obtain sufficient conditions on $T > 0$, $x_0 > 0$, $\Theta_0, \Theta_T \in L^2(\mathbb{R}^+)$ for the existence of exactly resolving controls. On the other hand, as the heat sources approaches the boundary ($x_0 \to 0$), a lack of exact controllability is observed (reported earlier by Micu and Zuazua [11]). Similarly, we derive sufficient conditions for the regularity of the infinite system characterizing exact controllability on $\mathbb{R}$.

Two sufficient conditions are obtained on $T > 0$, $\Theta_0, \Theta_T \in L^2(\Omega)$, for which the heat equation is approximately controllable in Ω. Moreover, it is obtained that, as $x_0 \to 0$, there is still possible to find such $T > 0$ and $\Theta_0 \in L^2(\mathbb{R}^+)$ for which the heat equation is approximately controllable with a required accuracy.

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